ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, No. 4, 1985

CLASSIFICATION OF FINITE GROUPS ACCORDING TO THE NUMBER OF CONJUGACY CLASSES

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ABSTRACT

We consider the problem of the classification of finite groups according to the number of conjugacy classes through the classification of all the finite groups with many minimal normal subgroups.

1. Introduction

In this work, G will denote a finite group, $r = r(G)$ the number of conjugacy classes, $\mathrm{Cl}_G(g_1) = \{1\}, \mathrm{Cl}_G(g_2), \ldots, \mathrm{Cl}_G(g_r)$ the conjugacy classes of G ordered so that $|\mathrm{Cl}_G(g_i)| \leqq |\mathrm{Cl}_G(g_{i+1})|$ for all $i = 1, \ldots, r$, $\Delta = \Delta_G = (|C_G(g_1)|, \ldots, |C_G(g_r)|)$, $\beta(G)$ the number of minimal normal subgroups of G and $\alpha(G)$ the number of conjugate classes of G not contained in the socle $S(G)$.

The possibility of classifying finite groups according to the number $r(G)$ and to some properties of their conjugacy classes was suggested in [2]. Around 1910, G. A. Miller and W. Burnside (cf. [2], Note A) derived those finite groups with $r(G) \leqq 5$. D. T. Sigley (cf. [19]) in 1935 studied those with $r(G) = 6$, and for $r(G) = 7$ he derived those with non-trivial centre, but his list for $r(G) = 6$ was incomplete. In 1966, J. Poland (cf. [17]) obtained all finite groups with $r(G) = 6$ and $r(G) = 7$.

In 1974 L. F. Kosvintsev (cf. [12]) classified all finite groups with exactly eight conjugate classes working with an I.B.M. and needing a large amount of calculations to select the valid solutions among those proposed by the computer.

In 1976 (cf. [16]) V. A. Odincov and A. I. Starostin got all finite groups with $r(G) = 9$.

Received July 1, 1984 and in revised form January 1, 1985

In 1978, A. G. Aleksandrov and K. A. Komissarcik (cf. [1]) classified all simple finite groups with $r(G) \leqq 12$.

Other works related to the above-mentioned problems are those of A. Mann (cf. [14]), W. Feit and J. Thompson (cf. [6]), M. Suzuki (cf. [20]) and the work of F. M. Markell (cf. [15]), where he classified the supersolvable groups with many conjugacy classes with different cardinality. He proved that if $|\mathrm{Cl}_G(g_i)| \neq |\mathrm{Cl}_G(g_j)|$ for every $i \neq j$ and G is supersolvable, then G is isomorphic to Σ_3.

In this paper, we approach the problem of the classification of finite groups according to the number $r(G)$ in a different way to the one used by the aforementioned mathematicians. We consider this problem through the classification of all the finite groups with many minimal normal subgroups.

Since each minimal normal subgroup is generated by the elements of a conjugacy class, our problem is equivalent to the classification of the families $\Phi_j = \{G \mid \beta(G) = r(G) - j\}$ for small values of the natural number j.

In this paper, the families Φ_i, $i = 1,2,\ldots,10$ are classified. Moreover, as an immediate corollary we find not only the previously known classification of finite groups with $r(G) \leqq 9$, but also that of those finite groups satisfying one of the following conditions:

(i) $r(G) = 10$,

(ii) $r(G) = 11$,

(iii) $r(G) = 12$ and $\beta(g) > 1$,

(iv) $r(G) = 13$ and $\beta(G) > 2$,

(v) $r(G) = 14$ and $\beta(G) > 3$,

(vi) $r(G) = n$ and $\beta(G) = n - a$ with $1 \leqq a \leqq 10$, for each integer $n \geqq 15$.

The analysis of the groups in Φ_j is done in three steps. First we study $\Psi_j = \Phi_j \cap \Gamma$, Γ being the class of finite nilpotent groups (cf. [22]). Afterwards we study the groups of $\Phi_j - \Psi_j$ with $S(G)$ non-solvable (Theorem 3.2) and finally we get all the groups in $\Phi_j - \Psi_j$ with $S(G)$ abelian, by fixing the number $\alpha(G)$ and studying the solutions of the associated equations (1) and (2) (Lemmas (2.1) and (2.2) with $N = S(G)$). These yield all possible structures of $S(G)$ and all possible actions by conjugation of $G/S(G)$ over $S(G)$, as well as the associated values of $r(G)$.

All reasoning corresponding to groups with different values of $r(G)$ and the same $\alpha(G)$ is no longer needed. Moreover, many of the r-tuples $(m_1,\ldots,m_r)$ that are solutions of the equation:

$$(*) \qquad 1 = \sum_{i=1}^{r} 1/m_i, \qquad m_i = |C_G(g_i)|$$

and for which no corresponding group exists do not show up in our analysis. This eliminates the sophisticated arguments used to show that those r-tuples do not have a corresponding group, simplifying all the proofs known up to now for the cases $r(G) \leqq 9$.

In this paper, the following notation will be used throughout:

$|A|$ denotes the cardinal of the set A,

$\dot{\cup}$ is the symbol of disjoint union,

C_p^m denotes an elementary abelian p-group of order p^m,

C_m denotes a cyclic group of order m,

Y denotes a direct product of elementary abelian groups, i.e., a group of the type $C_{p_1}^{m_1} \times \cdots \times C_{p_t}^{m_t}$,

H, a direct product of elementary abelian groups of odd orders,

$N \times_f K$, semidirect product of N and K with fixed-point-free action f (abbrev. f acts f.p.f.), i.e., a Frobenius's group of kernel N and complement K,

$N \times_\lambda K$, semidirect product of N and K, with N as a normal subgroup with respect to the action λ.

We use the standard notation for the non-abelian simple groups that appear in this work.

P_1 denotes a 2-group of type PSL(3,4), i.e., isomorphic to a Sylow 2-subgroup of PSL(3,4),

P_2, a 2-group of type PSU(3,4),

P_3, a 2-group of type Sz(8),

Q_1, a non-abelian 3-group of order 3^3 and exponent 3,

Q_2, a non-abelian 3-group of order 3^3 and exponent 9,

Σ_m, the symmetric group of degree m,

D_{2m}, the dihedral group of order $2m$,

Q_{2^m}, the quaternion group of order 2^m,

SD_{2^m}, the semidihedral group of order 2^m,

Hol G, holomorph of G,

Hol(G, K), relative holomorph of G and K,

C_2^3.PSL(2,7), the unique non-split extension of C_2^3 by PSL(2,7),

$(N_1 N_2)_K$, a direct product of subgroups N_1 and N_2 with amalgamated subgroup K,

$2^n\Gamma_i c_j$, j-th group of order 2^n of the family Γ_i with genus c_j (Hall–Senior's notation, cf. [22]),

$2^n\Gamma_i$, the set of groups of order 2^n of the family Γ_i (cf. [22]),

$SL(2,3)$. C_4, the unique extension of C_2 by Σ_4 which has Sylow 2-subgroups of type Q_{16}.

If S, T are subsets of G, we write S^T for the subset consisting of all conjugate elements $x^y = y^{-1}xy$ with $x \in S$, $y \in T$. Thus

$$S^T = \{x^y \mid x \in S, y \in T\}.$$

Also we define:

$F_{t,1} = C_2^t$, $F_{t,2} = C_3^t \times_f C_2$, $F_{t,3} = C_2^{2t} \times_f C_3$,
$F_{t,4} = C_5^t \times_f C_4 = (\Pi\langle a_i \rangle) \times_f \langle b \rangle$ with relations: $a_i^b = a_i^2$ for every i,
$F_{t,5} = C_7^t \times_f C_6 = (\Pi\langle a_i \rangle) \times_f \langle b \rangle$ with relations: $a_i^b = a_i^2$ for every i,
$F_{t,6} = C_2^{3t} \times_f C_7 = (\Pi\langle a_i \rangle \times \langle b_i \rangle \times \langle c_i \rangle) \times_f \langle d \rangle$ with relations: $a_i^d = b_i$, $b_i^d = c_i$, $c_i^d = a_ib_i$ for every i,
$F_{t,7} = C_3^{2t} \times_f C_8 = (\Pi\langle a_i \rangle \times \langle b_i \rangle) \times_f \langle c \rangle$ with relations: $a_i^c = b_i$, $b_i^c = a_ib_i$ for every i,
$F_{t,8} = C_{11}^t \times_f C_{10} = (\Pi\langle a_i \rangle) \times_f \langle b \rangle$ with relations: $a_i^b = a_i^2$ for every i.

Finally, if $\varnothing \neq S \subseteq G$, we define

$$r_G(S) = |\{\mathrm{Cl}_G(g) \mid \mathrm{Cl}_G(g) \cap S \neq \varnothing\}|.$$

Clearly, $r_G(S)$ is the number of conjugacy classes of G that compose the normal set $\bigcup_{g\in G} S^g$, and if $\bigcup_{g \in G} x^g S(G) = \bigcup_{i=1}^t \mathrm{Cl}_G(x_i)$ with $|C_G(x_j)| \leqq |C_G(x_{j+1})|$ (hence $t = r_G(xS(G))$), we define

$$\Delta_x = (|C_G(x_1)|, \dots, |C_G(x_t)|).$$

Now, the finite groups satisfying the conditions $r(G) \leqq 11$, and $r(G) = 12$, $\beta(G) > 1$, are described in Tables 1–4, which list the r-tuples $\Delta = (m_1, \dots, m_r)$ and the structures of $G/S(G)$.

2. Preliminaries

In the following, N is a normal subgroup of G, $\bar{G} = G/N$, $\mathrm{Cl}_{\bar{G}}(\bar{x}_1) = \{\bar{1}\}, \dots, \mathrm{Cl}_{\bar{G}}(\bar{x}_t)$ are the conjugacy classes of $\bar{G}$, and

$$T_{x_j} = \{g \in G \mid \mathrm{Cl}_G(g) \cap x_jN \neq \varnothing\} = \mathrm{Cl}_G(x_jn_1) \,\dot{\cup}\, \cdots \,\dot{\cup}\, \mathrm{Cl}_G(x_jn_{s_j})$$

with $n_k \in N$ and $s_j = r_G(x_jN)$. We have:

LEMMA 2.1. (i)

$$r_G(G - N) = \sum_{j=2}^{t} r_G(x_jN). \tag{1}$$

TABLE 1. The finite groups satisfying $r(G) \leqq 9$

$r(G)$	G	Δ	$G/S(G)$
1	$\{1\}$	(1)	1
2	C_2	(2,2)	1
	C_3	(3,3,3)	1
3	Σ_3	(6,3,2)	C_2
	C_4	(4,4,4,4)	C_2
	$C_2 \times C_2$	(4,4,4,4)	1
4	D_{10}	(10,5,5,2)	C_2
	A_4	(12,4,3,3)	C_3
	C_5	(5,5,5,5,5)	1
	D_8	(8,8,4,4,4)	$C_2 \times C_2$
	Q_8	(8,8,4,4,4)	$C_2 \times C_2$
5	D_{14}	(14,7,7,7,2)	C_2
	$C_5 \times_f C_4$	(20,5,4,4,4)	C_4
	$C_7 \times_f C_3$	(21,7,7,3,3)	C_3
	Σ_4	(24,8,4,4,3)	Σ_3
	A_5	(60,5,5,4,3)	1
	C_6	(6,6,6,6,6,6)	1
	$C_2 \times \Sigma_3$	(12,12,6,6,4,4)	C_2
	DC_3	(12,12,6,6,4,4)	C_2
	D_{18}	(18,9,9,9,9,2)	Σ_3
6	$C_3^2 \times_f C_2$	(18,9,9,9,9,2)	C_2
	$C_3^2 \times_f C_4$	(36,9,9,4,4,4)	C_4
	$C_3^2 \times_f Q_8$	(72,9,8,4,4,4)	Q_8
	PSL(2,7)	(168,8,7,7,4,3)	1
	C_7	(7,7,7,7,7,7,7)	1
	D_{16}	(16,16,8,8,8,4,4)	D_8
	Q_{16}	(16,16,8,8,8,4,4)	D_8
	SD_{16}	(16,16,8,8,8,4,4)	D_8
	D_{22}	(22,11,11,11,11,11,2)	C_2
	SL(2,3)	(24,24,6,6,6,6,4)	A_4
7	$C_{13} \times_f C_3$	(39,13,13,13,13,3,3)	C_3
	$C_7 \times_f C_6$	(42,7,6,6,6,6,6)	C_6
	$C_{13} \times_f C_4$	(52,13,13,13,4,4,4)	C_4
	$C_{11} \times_f C_5$	(55,11,11,5,5,5,5)	C_5
	Σ_5	(120,12,8,6,6,5,4)	C_2
	A_6	(360,9,9,8,5,5,4)	1
	C_8	(8,8,8,8,8,8,8,8)	C_4
	$C_2 \times C_4$	(8,8,8,8,8,8,8,8)	C_2
	$C_2 \times C_2 \times C_2$	(8,8,8,8,8,8,8,8)	1

TABLE 1 (contd.)

$r(G)$	G	Δ	$G/S(G)$
	$C_2\times D_{10}$	(20,20,10,10,10,10,4,4)	C_2
	$C_5\times_\lambda C_4$	(20,20,10,10,10,10,4,4)	C_2
	$C_2\times A_4$	(24,24,8,8,6,6,6,6)	C_3
	D_{26}	(26,13,13,13,13,13,13,2)	C_2
8	$C_2^4\times_f C_3$	(48,16,16,16,16,16,3,3)	C_3
	$C_4^2\times_f C_3$	(48,16,16,16,16,16,3,3)	A_4
	GL(2,3)	(48,48,8,8,8,6,6,4)	Σ_4
	SL(2,3). C_4	(48,48,8,8,8,6,6,4)	Σ_4
	$C_2^3\times_f C_7$	(56,8,7,7,7,7,7,7)	C_7
	$C_{17}\times_f C_4$	(68,17,17,17,17,4,4,4)	C_4
	$C_{13}\times_f C_6$	(78,13,13,6,6,6,6,6)	C_6
	$C_2^4\times_f C_5$	(80,16,16,16,5,5,5,5)	C_5
	Hol $(C_2^3, C_7\times_f C_3)$	(168,24,7,7,6,6,6,6)	$C_7\times_f C_3$
	$C_5^2\times_f Q_8$	(200,25,25,25,8,4,4,4)	Q_8
8	$C_5^2\times_f DC_3$	(300,25,25,12,6,6,4,4)	DC_3
	$C_5^2\times_f SL(2,3)$	(600,25,24,6,6,6,6,4)	SL(2,3)
	PSL(2,11)	(660,12,11,11,6,6,5,5)	1
	M_9 = PGL*(2,9)	(720,16,9,8,8,8,5,4)	C_2
	C_9	(9,9,9,9,9,9,9,9,9)	C_3
	$C_3\times C_3$	(9,9,9,9,9,9,9,9,9)	1
	$C_3\times \Sigma_3$	(18,18,18,9,9,9,6,6,6)	C_2
	$C_{12}\times_\lambda C_2$	(24,24,12,12,12,12,12,4,4)	C_2^2
	$C_3\times_\lambda D_8$	(24,24,12,12,12,12,12,4,4)	C_2^2
	$C_3\times_\lambda Q_8$	(24,24,12,12,12,12,12,4,4)	C_2^2
	D_{30}	(30,15,15,15,15,15,15,15,2)	C_2
	$\Sigma_3\times\Sigma_3$	(36,18,18,12,12,9,6,6,4)	C_2^2
	$C_{19}\times_f C_3$	(57,19,19,19,19,19,19,3,3)	C_3
	$C_{15}\times_\lambda C_4$	(60,30,15,15,15,12,6,4,4)	C_4
	$C_3^2\times_f C_8$	(72,9,8,8,8,8,8,8,8)	C_8
	$C_3^2\times_\lambda D_8$	(72,18,18,12,12,8,6,6,4)	D_8
	$C_2^2\times_\lambda D_{18}$	(72,36,24,12,9,9,9,4,4)	Σ_3
9	$(C_3\times A_4)\times_\lambda C_2$	(72,36,24,12,9,9,9,4,4)	Σ_3
	$C_{19}\times_f C_6$	(114,19,19,19,6,6,6,6,6)	C_6
	SL(2,5)	(120,120,10,10,10,10,6,6,4)	A_5
	Hol(C_3^2, SD_{16})	(144,18,16,12,8,8,8,6,4)	SD_{16}
	$P_1\times_f C_3$	(192,64,16,16,16,16,16,3,3)	$C_2^4\times_f C_3$
	$P_2\times_f C_3$	(192,64,16,16,16,16,16,3,3)	$C_2^4\times_f C_3$
	PGL(2,7)	(336,16,12,8,8,8,7,6,6)	C_2
	SL(2,8)	(504,9,9,9,9,8,7,7,7)	1
	$C_2^4\times_{\lambda_1} A_5$	(960,64,16,16,16,16,5,5,3)	A_5
	PSL(2,13)	(1092,13,13,12,7,7,7,6,6)	1
	$C_7^2\times_f$ SL(2,3)	(1176,49,49,24,6,6,6,6,4)	SL(2,3)
	$C_7^2\times_f$ SL(2,3). C_4	(2352,49,48,8,8,8,6,6,4)	SL(2,3)$\cdot C_4$
	A_7	(2520,36,24,12,9,7,7,5,4)	1

(ii) *If* $\bar{x}_j$ *is conjugate to* $\bar{y}_j$ *in* $\bar{G}$, *then* $r_G(x_jN) = r_G(y_jN)$.

(iii) $1/|C_{\bar{G}}(\bar{x}_j)| = \sum_{k=1}^{s_j} 1/|C_G(x_jn_k)|$ (cf. [16]) *and* $o(\bar{x}_j)$ *divides* $|C_G(x_jn_k)|$ *for every* $k = 1,\ldots,s_j$.

(iv) $r_G(x_jN) = 1$ *if and only if* $|C_G(x_j)| = |C_{\bar{G}}(\bar{x}_j)|$. *Furthermore in this case, we have* $x_jN \subseteq \mathrm{Cl}_G(x_j)$ (cf. [16]).

TABLE 2. The finite groups with exactly ten conjugacy classes

G	Δ	$G/S(G)$	Reference
C_{10}	(10, 10, 10, 10, 10, 10, 10, 10, 10, 10)	1	(2.17)
$C_2 \times D_8$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	C_2^2	(2.17)
$C_2 \times Q_8$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	C_2^2	(2.17)
$C_4 \times_\lambda C_4$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	C_2^2	(2.17)
$(C_4 \times C_2) \times_{\lambda_1} C_2$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	C_2^2	(2.17)
$C_8 \times_\lambda C_2$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	D_8	(2.17)
$(C_4 \times C_2) \times_{\lambda_2} C_2$	(16, 16, 16, 16, 8, 8, 8, 8, 8, 8)	C_2^3	(2.17)
$C_7 \times_\lambda C_4$	(28, 28, 14, 14, 14, 14, 14, 14, 4, 4)	C_2	(2.19)
$C_2 \times D_{14}$	(28, 28, 14, 14, 14, 14, 14, 14, 4, 4)	C_2	(2.19)
$C_{17} \times_f C_2$	(34, 17, 17, 17, 17, 17, 17, 17, 17, 2)	C_2	(2.18)
$C_5 \times_\lambda C_8$	(40, 40, 10, 10, 8, 8, 8, 8, 8, 8)	C_4	(4.2)
$C_2 \times \mathrm{Hol}\, C_5$	(40, 40, 10, 10, 8, 8, 8, 8, 8, 8)	C_4	(4.2)
$C_2 \times (C_7 \times_f C_3)$	(42, 42, 14, 14, 14, 14, 6, 6, 6, 6)	C_3	(4.1)
$C_2 \times \Sigma_4$	(48, 48, 16, 16, 8, 8, 8, 8, 6, 6)	Σ_3	(4.2)
$(C_2 \times C_2) \times_\lambda \mathrm{DC}_3$	(48, 48, 16, 16, 8, 8, 8, 8, 6, 6)	Σ_3	(4.2)
$Q_1 \times_{\lambda_1} C_2$	(54, 54, 54, 9, 9, 9, 9, 6, 6, 6)	$C_3^2 \times_f C_2$	(4.2)
$Q_1 \times_{\lambda_2} C_2$	(54, 27, 18, 18, 9, 9, 9, 6, 6, 6)	$C_3 \times \Sigma_3$	(4.11)
$Q_2 \times_\lambda C_2$	(54, 27, 18, 18, 9, 9, 9, 6, 6, 6)	$C_3 \times \Sigma_3$	(4.11)
$C_2^4 \times_{\lambda_1} \Sigma_3$	(96, 32, 32, 32, 16, 8, 8, 8, 8, 3)	Σ_3	(4.2)
$C_2^4 \times_{\lambda_2} C_6$	(96, 32, 24, 16, 16, 8, 6, 6, 6, 6)	$C_2 \times A_4$	(4.8)
$(C_4 \times C_4) \times_\lambda \Sigma_3$	(96, 32, 32, 32, 16, 8, 8, 8, 8, 3)	Σ_4	(4.2)
$(C_4 \times C_4) \times_\lambda C_6$	(96, 32, 24, 16, 16, 8, 6, 6, 6, 6)	$C_2 \times A_4$	(4.8)
$C_2^4 \times_{\lambda_2} C_6$	(96, 32, 24, 16, 16, 8, 6, 6, 6, 6)	$C_2 \times A_4$	(4.8)
$(C_5 \times C_5) \times_{f_1} C_4$	(100, 25, 25, 25, 25, 25, 25, 4, 4, 4)	C_4	(2.20)
$(C_5 \times C_5) \times_{f_2} C_4$	(100, 25, 25, 25, 25, 25, 25, 4, 4, 4)	C_4	(2.20)
$(C_5 \times C_5) \times_{f_3} C_4$	(100, 25, 25, 25, 25, 25, 25, 4, 4, 4)	C_4	(2.20)
$C_{25} \times_f C_4$	(100, 25, 25, 25, 25, 25, 25, 4, 4, 4)	$\mathrm{Hol}\, C_5$	(4.2)
$C_2 \times A_5$	(120, 120, 10, 10, 10, 10, 8, 8, 6, 6)	1	(3.2)
$C_{17} \times_f C_8$	(136, 17, 17, 8, 8, 8, 8, 8, 8, 8)	C_8	(4.8)
$(C_5 \times C_5) \times_f C_6$	(150, 25, 25, 25, 25, 6, 6, 6, 6, 6)	C_6	(4.2)
$C_2^4 \times_\lambda D_{10}$	(160, 32, 32, 32, 8, 8, 8, 8, 5, 5)	D_{10}	(4.2)
$\mathrm{Hol}(C_3^2, \mathrm{SL}(2,3))$	(216, 27, 18, 18, 9, 9, 24, 6, 6, 4)	$\mathrm{SL}(2,3)$	(4.5)
$P_3 \times_f C_7$	(448, 64, 16, 16, 7, 7, 7, 7, 7, 7)	$C_2^3 \times_f C_7$	(4.8)
$(C_7 \times C_7) \times_f \mathrm{DC}_3$	(588, 49, 49, 49, 49, 12, 6, 6, 4, 4)	DC_3	(4.2)
$(C_7 \times C_7) \times_f Q_{16}$	(784, 49, 49, 49, 16, 8, 8, 8, 4, 4)	Q_{16}	(4.5)
M_{11}	(7920, 48, 18, 11, 11, 8, 8, 8, 6, 5)	1	(3.2)
$(C_{11} \times C_{11}) \times_f \mathrm{SL}(2,5)$	(14520, 121, 120, 10, 10, 10, 10, 6, 6, 4)	$\mathrm{SL}(2,5)$	(4.11)
$\mathrm{PSL}(3,4)$	(20160, 64, 16, 16, 16, 9, 7, 7, 5, 5)	1	(3.2)

PROOF. (i) We have $\mathrm{Cl}_G(g) \subseteq G - N$ if and only if $\bar{g}$ is conjugate to $\bar{x}_j$ for some $j \geqq 2$, i.e. if $\mathrm{Cl}_G(g) \cap x_jN \neq \varnothing$ for some $j \geqq 2$. Moreover,

$$(\mathrm{Cl}_G(g) \cap x_jN) \cap (\mathrm{Cl}_G(g) \cap x_1N) = \varnothing \qquad \text{for every } j \neq 1,$$

because $\bar{x}_j$ is not conjugate to $\bar{x}_1$ in $\bar{G}$, hence $r_G(G-N) = \Sigma_{j=2}^{t} r_G(x_jN)$.

(ii) If $\bar{x}_j$ and $\bar{y}_j$ are conjugate in $\bar{G}$, then x_jN and y_jN are conjugate in G, hence $r_G(x_jN) = r_G(y_jN)$.

TABLE 3. The finite groups with exactly eleven conjugacy classes

G	Δ	$G/S(G)$	Reference
C_{11}	(11,11,11,11,11,11,11,11,11,11,11)	1	(2.17)
Q_1	(27,27,27,9,9,9,9,9,9,9,9)	$C_3 \times C_3$	(2.17)
Q_2	(27,27,27,9,9,9,9,9,9,9,9)	$C_3 \times C_3$	(2.17)
D_{32}	(32,32,16,16,16,16,16,16,16,4,4)	D_{16}	(2.17)
Q_{32}	(32,32,16,16,16,16,16,16,16,4,4)	D_{16}	(2.17)
SD_{32}	(32,32,16,16,16,16,16,16,16,4,4)	D_{16}	(2.17)
$2^5\Gamma_6 a_1$	(32,32,16,16,16,8,8,8,8,8,8)	$C_2 \times D_8$	(2.17)
$2^5\Gamma_6 a_2$	(32,32,16,16,16,8,8,8,8,8,8)	$C_2 \times D_8$	(2.17)
$2^5\Gamma_7 a_1$	(32,32,16,16,16,8,8,8,8,8,8)	$(C_4 \times C_2) \times_{\lambda_1} C_2$	(2.17)
$2^5\Gamma_7 a_2$	(32,32,16,16,16,8,8,8,8,8,8)	$(C_4 \times C_2) \times_{\lambda_1} C_2$	(2.17)
$2^5\Gamma_7 a_3$	(32,32,16,16,16,8,8,8,8,8,8)	$(C_4 \times C_2) \times_{\lambda_1} C_2$	(2.17)
$C_{19} \times_f C_2$	(38,19,19,19,19,19,19,19,19,19,2)	C_2	(2.18)
$(C_5 \times C_5) \times_f C_3$	(75,25,25,25,25,25,25,25,25,3,3)	C_3	(2.19)
$((Q_8 Q_8)_{C_2}) \times_\lambda C_3$	(96,96,16,16,16,16,16,6,6,6,6)	$C_2^4 \times_f C_3$	(4.8)
$Q_1 \times_\lambda (C_2 \times C_2)$	(108,54,18,18,12,12,12,9,6,6,6)	$\Sigma_3 \times \Sigma_3$	(4.11)
$C_{11} \times_f C_{10}$	(110,11,10,10,10,10,10,10,10,10,10)	C_{10}	(4.14)
$\dot{C}_{29} \times_f C_4$	(116,29,29,29,29,29,29,29,4,4,4)	C_4	(2.20)
$C_{31} \times_f C_5$	(155,31,31,31,31,31,31,5,5,5,5)	C_5	(4.1)
$C_{19} \times_f C_9$	(171,19,19,9,9,9,9,9,9,9,9)	C_9	(4.11)
$C_{31} \times_f C_6$	(186,31,31,31,31,31,6,6,6,6,6)	C_6	(4.2)
$C_2^4 \times_\lambda DC_3$	(192,64,48,16,16,8,8,8,8,6,6)	$(C_2 \times C_2) \times_\lambda \mathrm{DC}_3$	(4.14)
$(C_4 \times C_4) \times_\lambda \mathrm{DC}_3$	(192,64,48,16,16,8,8,8,8,6,6)	$(C_2 \times C_2) \times_\lambda \mathrm{DC}_3$	(4.14)
$(C_5 \times C_5) \times_f C_8$	(200,25,25,25,8,8,8,8,8,8,8)	C_8	(4.8)
$C_{29} \times_f C_7$	(203,29,29,29,29,7,7,7,7,7,7)	C_7	(4.5)
$C_2^4 \times_\lambda \mathrm{Hol}\, C_5$	(320,64,32,16,16,8,8,8,8,8,5)	$\mathrm{Hol}\, C_5$	(4.2)
SL(2,7)	(336,336,14,14,14,14,8,8,8,6,6)	PSL(2,7)	(4.2)
$(C_7 \times C_7) \times_f Q_8$	(392,49,49,49,49,49,49,8,4,4,4)	Q_8	(4.1)
$\mathrm{Hol}(C_3 \times C_3)$	(432,54,48,18,12,9,8,8,8,6,6)	GL(2,3)	(4.8)
PGL(2,9)	(720,20,16,10,10,10,10,9,8,8,8)	C_2	(3.2)
Σ_6	(720,48,48,18,18,16,8,8,6,6,5)	C_2	(3.2)
$\mathrm{Hol}(C_2^3)$	(1344,192,32,32,16,8,8,7,7,6,6)	PSL(2,7)	(4.2)
$C_2^3 \cdot \mathrm{PSL}(2,7)$	(1344,192,32,32,16,8,8,7,7,6,6)	PSL(2,7)	(4.2)
PΓL(2,8)	(1512,27,24,18,18,9,9,9,7,6,6)	C_3	(3.2)
PSL(2,17)	(2448,17,17,16,9,9,9,9,8,8,8)	1	(3.2)
Sz(8)	(29.120,64,16,16,13,13,13,7,7,7,5)	1	(3.2)

(iii) We have $T_{x_j} = \bigcup_{g\in G} x_j^g N$, hence $|\mathrm{Cl}_{\bar G}(\bar x_j)|\cdot|N| = \Sigma_{k=1}^{s_j}|\mathrm{Cl}_G(x_j n_k)|$, therefore $1/|C_{\bar G}(\bar x_j)| = \Sigma_{k=1}^{s_j} 1/|C_G(x_j n_k)|$. Moreover $o(\bar x_j) = o(x_j n_k)$ divides $|C_G(x_j n_k)|$ for every k.

(iv) It is an immediate consequence of (ii).

LEMMA 2.2. *Suppose that N is abelian. We have:*

(i) If $x, y \in T_{x_j}$, then $|C_G(x)\cap N| = |C_G(y)\cap N|$ and $o(\bar x_j)\cdot|C_G(x_j)\cap N|$ is a divisor of $|C_G(x_j n_k)|$ for every k. Further, if $\bar x_j \in \langle \bar z\rangle$, then there exists $z^e \in T_{x_j}$ such that $o(\bar z)\cdot|C_G(x_j)\cap N|$ divides $|C_G(z^e)|$.

(ii) $r_G(x_j N) \leqq |C_G(x_j)\cap N|$ and the equality holds if and only if $\overline{C_G(x_j w)} = C_{\bar G}(\bar x_j)$ for every $w \in N$, where $\overline{C_G(x_j w)} = C_G(x_j w)N/N$.

TABLE 4. The finite groups satisfying $r(G) = 12$ and $\beta(G) > 1$

G	Δ	$G/S(G)$	Reference
$C_2\times C_6$	(12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12)	1	(2.17)
C_{12}	(12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12)	C_2	(2.17)
$C_3\times_\lambda C_8$	(24, 24, 24, 24, 12, 12, 12, 12, 8, 8, 8, 8)	C_4	(4.2)
$C_4\times\Sigma_3$	(24, 24, 24, 24, 12, 12, 12, 12, 8, 8, 8, 8)	C_2^2	(4.2)
$C_2^2\times\Sigma_3$	(24, 24, 24, 24, 12, 12, 12, 12, 8, 8, 8, 8)	C_2	(4.1)
$C_2\times DC_3$	(24, 24, 24, 24, 12, 12, 12, 12, 8, 8, 8, 8)	C_2	(4.1)
$C_3\times D_{10}$	(30, 30, 30, 15, 15, 15, 15, 15, 15, 6, 6, 6)	C_2	(2.20)
$C_2\times(C_9\times_f C_2)$	(36, 36, 18, 18, 18, 18, 18, 18, 18, 18, 4, 4)	Σ_3	(4.2)
$C_9\times_\lambda C_4$	(36, 36, 18, 18, 18, 18, 18, 18, 18, 18, 4, 4)	Σ_3	(4.2)
$C_3\times A_4$	(36, 36, 36, 12, 12, 12, 9, 9, 9, 9, 9, 9)	C_3	(4.2)
$C_3^2\times_\lambda C_4$	(36, 36, 18, 18, 18, 18, 18, 18, 18, 18, 4, 4)	C_2	(2.19)
$C_2\times(C_3^2\times_f C_2)$	(36, 36, 18, 18, 18, 18, 18, 18, 18, 18, 4, 4)	C_2	(2.19)
$C_2^2\times_\lambda C_9$	(36, 36, 36, 12, 12, 12, 9, 9, 9, 9, 9, 9)	C_3	(4.2)
$(C_3\times C_7)\times_f C_2$	(42, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 2)	C_2	(2.18)
$C_3\times_\lambda Q_{16}$	(48, 48, 24, 24, 24, 12, 12, 12, 12, 8, 8, 4)	D_8	(4.2)
$C_3\times_\lambda D_{16}$	(48, 48, 24, 24, 24, 12, 12, 12, 12, 8, 8, 4)	D_8	(4.2)
$C_3\times_\lambda \mathrm{SD}_{16}$	(48, 48, 24, 24, 24, 12, 12, 12, 12, 8, 8, 4)	D_8	(4.2)
$\Sigma_3\times D_{10}$	(60, 30, 30, 30, 20, 15, 15, 12, 10, 10, 6, 4)	C_2^2	(4.2)
$C_3^2\times_\lambda C_8$	(72, 72, 18, 18, 18, 18, 8, 8, 8, 8, 8, 8)	C_4	(4.2)
$\Sigma_3\times A_4$	(72, 36, 24, 24, 18, 18, 12, 9, 9, 8, 6, 6)	C_6	(4.2)
$C_2\times(C_3^2\times_f C_4)$	(72, 72, 18, 18, 18, 18, 8, 8, 8, 8, 8, 8)	C_4	(4.2)
$(C_2^2\times C_7)\times_f C_3$	(84, 28, 28, 28, 28, 28, 28, 28, 28, 28, 3, 3)	C_3	(2.19)
$(C_2^2\times Q_8)\times_\lambda C_3$	(96, 96, 32, 32, 16, 16, 16, 16, 6, 6, 6, 6)	A_4	(4.2)
$C_3^3\times_\lambda C_4$	(108, 54, 27, 27, 27, 27, 27, 27, 12, 6, 4, 4)	C_4	(4.1)
$(C_3\times C_7)\times_\lambda C_6$	(126, 63, 21, 21, 21, 18, 18, 9, 9, 6, 6, 6)	C_6	(4.2)
$C_2\times(C_3^2\times_f Q_8)$	(144, 144, 18, 18, 16, 16, 8, 8, 8, 8, 8, 8)	Q_8	(4.2)
$C_3^2\times_\lambda(C_4\times_\lambda C_4)$	(144, 144, 18, 18, 16, 16, 8, 8, 8, 8, 8, 8)	Q_8	(4.2)
$\mathrm{Hol}(2^5\Gamma_2 h, C_3)$	(96, 96, 32, 32, 16, 16, 16, 16, 6, 6, 6, 6)	A_4	(4.2)
$(C_2^2\times C_7)\times_\lambda C_6$	(168, 56, 28, 28, 28, 28, 24, 8, 6, 6, 6, 6)	C_6	(4.2)
$C_3^3\times_\lambda Q_8$	(216, 108, 27, 27, 27, 24, 12, 12, 12, 12, 4, 4)	Q_8	(4.2)
$\mathrm{PSL}(2,7)\times C_2$	(336, 336, 16, 16, 14, 14, 14, 14, 8, 8, 6, 6)	1	(3.2)
$(A_5\times C_3)\times_\lambda C_2$	(360, 180, 24, 18, 15, 15, 15, 12, 12, 9, 6, 4)	C_2	(2.20)

(iii) *If* $C_{\bar G}(\bar x_j) = \langle \bar x_j \rangle$, *then* $r_G(x_jN) = |C_G(x_j) \cap N|$ *and* $|C_G(x_jn_k)| = |C_G(x_j)| = o(\bar x_j) \cdot |C_G(x_j) \cap N|$.

(iv) *If* $|C_{\bar G}(\bar x_j)| = p^{e_j}$, *with p a prime number, then there exist natural numbers* t_k *such that* $|C_G(x_jn_k)| = p^{t_k} \cdot m$ *with* $m = |O_{p'}(C_G(x_j) \cap N)|$, *for every* $k = 1, \dots, s_j$. *Further, we have the equation*

$$1/p^{e_j} = (1/m) \cdot \left(\sum_{k=1}^{s_j} 1/p^{t_k} \right); \tag{2}$$

$o(\bar x_j)$ *divides* p^{t_k} *for every k, and m divides* $|N|$.

(v) *If* $o(\bar x_j) = p^t$, *p a prime number and* $p \nmid |C_G(x_j) \cap N|$, *then* $p \nmid |N|$.

PROOF. (i) If $x, y \in T_{x_j}$, then there are $g \in G$ and $w \in N$ such that $x = y^g w$, hence $C_G(x) \cap N = (C_G(y) \cap N)^g$ because N is abelian. Consequently, $|C_G(x) \cap N| = |C_G(y) \cap N|$. Moreover, $\langle x_jn_k \rangle\, (C_G(x_jn_k) \cap N)$ is a subgroup of $C_G(x_jn_k)$ of order $o(\bar x_j) \cdot |C_G(x_j) \cap N|$ and if $\bar x_j \in \langle \bar z \rangle$, then there is an $u \in N$ such that $x_j = z^e u$ for some integer e, hence $z^e \in T_{x_j}$ and clearly $o(\bar z) \cdot |C_G(x_j) \cap N|$ divides $|C_G(z^e)|$.

(ii) We have

$$|C_G(x_jn_k)| = |C_G(x_jn_k) \cap N|\, |\overline{C_G(x_jn_k)}| = |C_G(x_j) \cap N|\, |\overline{C_G(x_jn_k)}|,$$

so the relation (1) yields

$$1/|C_{\bar G}(\bar x_j)| = (1/|C_G(x_j) \cap N|) \cdot \left(\sum_{k=1}^{s_j} 1/|\overline{C_G(x_jn_k)}| \right)$$

hence

$$|C_G(x_j) \cap N| = \sum_{k=1}^{s_j} |C_{\bar G}(\bar x_j)| / |\overline{C_G(x_jn_k)}|. \tag{3}$$

Also, $|\overline{C_G(x_jn_k)}| \leqq |C_{\bar G}(\overline{x_jn_k})| = |C_{\bar G}(\bar x_j)|$. Thus (3) implies $|C_G(x_j) \cap N| \geqq s_j$ and the equality holds if and only if $\overline{C_G(x_jw)} = C_{\bar G}(\bar x_j)$ for every $w \in N$.

(iii) It follows from (ii).

(iv) Let Q be a q-Sylow subgroup of $C_G(x_j)$ with $q \neq p$ a prime number. Since $C_{\bar G}(\bar x_j)$ is a p-group, Q is a subgroup of N, hence Q is a Sylow q-subgroup of $C_G(x_j) \cap N$. Thus, we have $|C_G(x_j)| = p^{t_1} m$ with $m = |O_{p'}(C_G(x_j) \cap N)|$ because $C_G(x_j) \cap N$ is abelian. Similarly, $C_{\bar G}(\overline{x_jn_k}) = C_{\bar G}(\bar x_j)$ and $C_G(x_j) \cap N = C_G(x_jn_k) \cap N$ imply that there exists a natural number t_k such that $|C_G(x_jn_k)| = p^{t_k} \cdot m$. This yields the relation (2).

(v) Set $o(\bar x_j) = p^t$, $x_j = z_1 z_2$ with $[z_1, z_2] = 1$, $o(z_1) = p^s$ and $p \nmid o(z_2)$. Then

$x_j^{p^s} \in N$ and $x_j^{p^s} = z_2^{p^s}$, hence $z_2 \in N$ and $C_G(x_j) \cap N = C_G(z_1) \cap N$, because N is abelian. Thus if p divides $|N|$ and P is a p-Sylow subgroup of G containing z_1, then $1 \neq Z(P) \cap O_p(N) \leqq C_G(z_1) \cap N$, hence p divides $|C_G(x_j) \cap N|$.

REMARK. We will use Lemma (2.2) with $N = S(G)$ abelian. Once the structure of $G/S(G)$ is fixed, we consider the tuple $\Delta_{\bar{G}}$. Then the relation (2) yields all the possible solutions of Δ_{x_j} for each $\bar{x}_j$ such that $C_{\bar{G}}(\bar{x}_j)$ is a p-group. For example, if $r_G(x_j S(G)) = 2$ and $|C_{\bar{G}}(\bar{x}_j)| = p^{e_j}$, then we have

$$1/p^{e_j} = (1/m) \cdot (1/p^{t_1} + 1/p^{t_2}) \qquad \text{with } t_1 \leqq t_2,$$

so either ($p = 2$, $t_1 = t_2$, $m = 1$ and $e_j = t_2 - 1$), or ($e_j = t_2$, $m = 1 + p^{t_2 - t_1}$, and $s \leqq t_1 \leqq e_j = t_2$, if $o(x_j) = p^s$). Thus with the fixed structure of $G/S(G)$, equations (1) and (2) will be used to obtain the number $r(G)$ and the tuple Δ_G, once the structure of $S(G)$ and the action by conjugation of $G/S(G)$ over $S(G)$ have been determined.

LEMMA 2.3. *Let N be a normal subgroup of G and $\bar{G} = G/N$. Then*
(i) $|C_{\bar{G}}(\bar{x})| \leqq |C_G(x)|$ *for each* $x \in G$.
(ii) *If* $r(\bar{G}) = r_G(G - N) + 1$, *then* $|C_G(x)| = |C_{\bar{G}}(\bar{x})|$ *for each* $x \in G - N$ *and consequently either G is a Frobenius group with kernel N or there exists a prime p dividing both $|G/N|$ and $|N|$ such that either N or G/N is a p-group. Furthermore, if N is a p-group and G is neither a p-group nor a Frobenius group of kernel N, then $O_{p'}(G/N) = \{\bar{1}\}$ and each p'-subgroup T of G acts* f.p.f. *on N. In particular, every Sylow subgroup of T is either cyclic or a generalized quaternion group and T does not possess any subgroup of type $C_{q_1} \times_f C_{q_2}$ with primes $q_1 \neq q_2$.*

PROOF. (i) Follows from Lemma (2.1)(iii).
(ii) Set

$$G = N \dot{\cup} \mathrm{Cl}_G(y_1) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_G(y_t). \tag{4}$$

By hypothesis, we have

$$\bar{G} = \{\bar{1}\} \dot{\cup} \mathrm{Cl}_{\bar{G}}(\bar{y}_1) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_{\bar{G}}(\bar{y}_t). \tag{5}$$

As $|\mathrm{Cl}_G(y_i)| \leqq |\mathrm{Cl}_{\bar{G}}(\bar{y}_i)| \cdot |N|$ for every i, by taking cardinals in (4) and (5) we deduce that $|C_G(y_i)| = |C_{\bar{G}}(\bar{y}_i)|$, hence G satisfies (F2) and the result holds (cf. [3] and [4]).

(The authors wish to acknowledge the referee's remarks on Lemma 2.3(ii).)

LEMMA 2.4. *Let T be a non-abelian 2-group of order 2^n. We have:*

(i) *There exists* $g \in T$ *such that* $|C_T(g)| = 4$ *if and only if* T *is isomorphic to one of the following groups*: D_{2^n}, Q_{2^n} *or* SD_{2^n}. *Furthermore, in this case, there is an* $a \in T$ *such that* $T/\langle a \rangle \simeq C_2$ *and* T *does not have any subgroup isomorphic to* C_2^m *with* $m \geqq 3$.

(ii) *If there exists* $g \in T$ *such that* $|C_T(g)| = 8$, *then* T *has sectional rank at most* 4, *hence* T *does not have any subgroup isomorphic to* C_2^m *with* $m \geqq 5$. (cf. [7] and [9]).

Lemma 2.5. *Let* G *be a finite group with* $S(G)$ *abelian. We have*:

(i) *If there exists* $g \in G$ *such that* $|C_G(g)| = 4m$ *with* m *an odd number and* $o(g)$ *is a power of* 2, *then* $O_2(S(G)) \leqq C_2^2$.

(ii) *If there exists* $g \in G$ *such that* $|C_G(g)| = 8m$ *with* m *an odd number and* $o(g)$ *is a power of* 2, *then* $O_2(S(G)) \leqq C_2^4$.

Proof. This result is an immediate consequence of (2.4).

Lemma 2.6 (Galois). *Let* G *be a solvable group such that* $\beta(G) = 1$ *and* $C_G(S(G)) = S(G)$. *Then* $S(G)$ *has a complement in* G *and all the complements of* $S(G)$ *in* G *are conjugates* (cf. [10, th. 3.3, p. 160]).

Lemma 2.7. *Let* $N \trianglelefteq G$ *be such that* N *is a product of some minimal normal subgroups of* G. *Then, if* K *is a normal subgroup of* G *such that* $K \leqq N$, *there exists* $E \trianglelefteq G$ *such that* $N = K \times E$ (cf. [18 Ex. 414, p. 171]).

Lemma 2.8. *Let* $A \trianglelefteq G$ *be such that* A *is abelian and* g.c.d. $(|G/A|, |A|) = 1$. *If* B *is a direct factor of* A *and* $B \trianglelefteq G$, *then there exists* $D \trianglelefteq G$ *such that* $A = B \times D$ (cf. [11, Ex. 20.2.5, p. 145]).

Lemma 2.9. *Let* $N \trianglelefteq G$ *be such that* $G/N \simeq C_p$, *with* p *a prime number, and set* $g \in G - N$. *Then*

(i) $r(G) = ps + ((r(N) - s)/p)$, *where* s *is the number of conjugate classes of* N *fixed by the conjugation automorphism* α_g *of* N *induced by* g.

(ii) $r_G(G - N) = s \cdot (p-1)$. *Furthermore, if* N *is abelian, then* $s = |C_G(g) \cap N|$.

(iii) *Let* R_i *be the set of conjugate classes of* N *of cardinality* m_i, $i = 1, \ldots, e$, *where* $\{m_1, \ldots, m_e\}$ *is the set of different cardinals of the conjugate classes of* N, *and set* s_i *the number of classes of* R_i *fixed by* α_g. *Then* $r(N) = \Sigma_{i=1}^{e} |R_i|$, $|R_i| \equiv s_i \pmod p$ *and* $s = \Sigma_{i=1}^{e} s_i$.

(iv) *Let* S_j *be the set of conjugate classes of* N *of elements of orders* m'_j, $j = 1, \ldots, e'$, *where* $\{m'_1, \ldots, m'_{e'}\}$ *is the set of different orders of the elements of* N, *and set* s'_j *the number of classes of* S_j *fixed by* α_g. *Then* $r(N) = \Sigma_{j=1}^{e'} |S_j|$, $|S_j| \equiv s'_j \pmod p$ *and* $s = \Sigma_{j=1}^{e'} s'_j$.

(v) *The subgroup* $\langle g\rangle$ *acts by conjugation over the set* $\mathrm{Irr}(N)$ *of all the complex irreducible characters of* N *and* $s=|\{\chi\in\mathrm{Irr}(N)\mid \chi^g=\chi\}|$. *Further, if* $\{m''_1,\dots,m''_{e''}\}$ *is the set of different degrees of the characters of* $\mathrm{Irr}(N)$, $X_l=\{\chi\in\mathrm{Irr}(N)\mid\chi(1)=m''_l\}$ *and* s''_l *is the number of the characters of* X_l *fixed by* g, *then* $r(N)=\Sigma_{l=1}^{e''}|X_l|$, $|X_l|\equiv s''_l \pmod p$ *and* $s=\Sigma_{l=1}^{e''}s''_l$.

PROOF. (i) It follows from [2] p. 472.

(ii) We have $r(G)=ps+(r(N)-s)/p$ and $r_G(N)=s+(r(N)-s)/p$ hence $r_G(G-N)=s\cdot(p-1)$. Moreover, if N is abelian, clearly, $s=|C_G(g)\cap N|$.

(iii) Since $|\mathrm{Cl}_N(n)^g|=|\mathrm{Cl}_N(n)|$ and $o(n^g)=o(n)$ for each conjugate class $\mathrm{Cl}_N(n)$ of N, clearly, (iii) and (iv) follow. On the other hand, by considering the action of $\langle g\rangle$ on $\mathrm{Irr}(N)$, Brauer's Lemma on character table (cf. [10 Satz 13.5]) yields $s=|\{\chi\in\mathrm{Irr}(N)\mid\chi^g=\chi\}|$.

(v) It holds because $\chi^g(1)=\chi(1)$ for each $\chi\in\mathrm{Irr}(N)$.

REMARK. Suppose that the tuples $(|\mathrm{Cl}_N(n_1)|,\dots,|\mathrm{Cl}_N(n_t)|)$, $(o(n_1),\dots,o(n_t))$ and $(\chi_1(1),\dots,\chi_t(1))$ with $t=r(N)$ are known. Then the Lemma 2.9 may be used to set bounds to the possible values of $r(G)$ and also to determine Δ_G.

LEMMA 2.10. *Let* N *be an abelian normal subgroup of* G *and let* K *be a subgroup of* G *such that* $G=NK$. *Suppose that there is an* $h\in K$ *satisfying*

$$C_{\bar K}(\bar h)=\overline{C_K(h)},\qquad \text{where } \overline{C_K(h)} \text{ is the image of } C_K(h) \text{ in } \bar G=G/N.$$

Then the following propositions hold.

(1) *If* $[h,N]=\{[h,n]\mid n\in N\}$, *then* $[h,N]$ *is a normal subgroup of* $B=NC_K(h)$ *and* $r_G(hN)=r_E(\tilde N)$, *where* $\tilde N=N/[h,N]$ *and* $E=B/[h,N]$.

(2) *If* $\{\tilde m_1=1,\tilde m_2,\dots,\tilde m_s\}$ *is a complete system of representatives from distinct conjugacy classes of* E *that make up the normal subgroup* $\tilde N$, *ordered so that*

$$|C_D(\tilde m_i)|\geqq|C_D(\tilde m_{i+1})|,\quad i=1,\dots,s-1,\qquad \text{where } D=(C_K(h)[h,N])/[h,N],$$

then,

(a) $|C_N(h)|=1+\Sigma_{i=2}^{s}|D:C_D(\tilde m_i)|$,

(b) $\Delta_h=(|C_N(h)|\cdot(|C_K(h)|/|D:C_D(\tilde m_s)|),\dots,$
$|C_N(h)|\cdot(|C_K(h)|/|D:C_D(\tilde m_2)|),|C_N(h)|\cdot|C_K(h)|)$

and $\mathrm{Cl}_G(hm_1),\dots,\mathrm{Cl}_G(hm_s)$ *is the set of all conjugacy classes of* G *that make up the normal set* $(hN)^G$.

PROOF. Certainly, $[h,N]$ is a normal subgroup of $B=NC_K(h)$, hence $\tilde N$ is an abelian normal subgroup of E. Let $\{z_1=1,z_2,\dots,z_t\}$ be a right transversal of B

in G. We can suppose that each z_i is an element of K. Since $\overline{C_K(h)} = C_K(h)N/N = C_{\bar{K}}(\bar{h})$, we have

$$\bar{G} = \dot{\bigcup_{i=1}^{t}} C_{\bar{K}}(\bar{h})\bar{z}_i,$$

i.e., $\{\bar{z}_1, \ldots, \bar{z}_t\}$ is a right transversal of $C_{\bar{K}}(\bar{h})$ in $\bar{G}$. For each $m \in N$ we have

$$\mathrm{Cl}_G(hm) = (hm)^G = \bigcup_{i=1}^{t} (hm)^{Bz_i}.$$

Moreover $(hm)^{Bz_i} \cap (hm)^{Bz_j} \neq \varnothing$ implies that there are elements $b, b' \in B$ such that $(hm)^{bz_i} = (hm)^{b'z_j}$, hence $\bar{h}^{\bar{z}_i} = \bar{h}^{\bar{z}_j}$, and necessarily $i = j$. Thus

$$\mathrm{Cl}_G(hm) = \dot{\bigcup_{i=1}^{t}} (hm)^{Bz_i} \quad \text{and} \quad |\mathrm{Cl}_G(hm)| = |\mathrm{Cl}_{\bar{G}}(\bar{h})| \cdot |(hm)^B|.$$

On the other hand, we have

$$(hm)^B = (hm)^{NC_K(k)} = (h[h,N]m)^{C_K(h)} = h[h,N]m^{C_K(h)},$$

so $|(hm)^B| = |[h,N]m^{C_K(h)}|$, but the set $[h,N]m^{C_K(h)}$ is a disjoint union of $|D : C_D(\tilde{m})|$ (with $\tilde{m} = m[h,N]$) right cosets of $[h,N]$ in N, hence

$$|(hm)^B| = |[h,N]| \cdot |D : C_D(\tilde{m})|. \tag{6}$$

Therefore $|\mathrm{Cl}_G(hm)| = |\mathrm{Cl}_{\bar{G}}(\bar{h})| \cdot |[h,N]| \cdot |D : C_D(\tilde{m})|$ and

$$\begin{aligned} |C_G(hm)| &= (|N/[h,N]|) \cdot (|C_K(h)|/|D : C_D(\tilde{m})|) \\ &= |C_N(h)| \cdot (|C_K(h)|/|D : C_D(\tilde{m})|). \end{aligned} \tag{7}$$

Furthermore, if m and m' are two elements of N, we note that $(hm)^B = (hm')^B$ if and only if $[h,N]m$ is conjugate to $[h,N]m'$ in E, because

$$(hm)^B = h[h,N]m^{C_K(h)}, \qquad (hm')^B = h[h,N]m'^{C_K(h)}$$

and $\tilde{N}$ is an abelian subgroup of E.

Let $\mathrm{Cl}_G(hm_1), \ldots, \mathrm{Cl}_G(hm_s)$ be the totality of conjugacy classes of G that make up $(hN)^G$, ordered so that $|C_G(hm_j)| \geqq |C_G(hm_{j+1})|$ for all $j = 1, 2, \ldots, s-1$. We have $\mathrm{Cl}_G(hm_j) \cap hN = (hm_j)^B$ (because $(hm_j)^{Bz_i} \cap hN \neq \varnothing$ implies $\bar{h}^{\bar{z}_i} = \bar{h}$, hence $\bar{z}_i = \bar{1}$) and

$$hN = \dot{\bigcup_{j=1}^{s}} (\mathrm{Cl}_G(hm_j) \cap hN) = \dot{\bigcup_{j=1}^{s}} (hm_j)^B.$$

Counting the number of elements of H in both sides of the above equality, we get (2)(a) from the relation (7). Also, (6) yields the proposition (2)(b).

Finally, we have $\mathrm{Cl}_G(hm_j) = \mathrm{Cl}_G(hm_{j'})$ if and only if $(hm_j)^B = \mathrm{Cl}_G(hm_j) \cap hN = \mathrm{Cl}_G(hm_{j'}) \cap hN = (hm_{j'})^B$, i.e. iff $\tilde{m}_j$ is conjugate to $\tilde{m}_{j'}$ in E. Therefore $\{\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_s\}$ is a complete system of representatives from distinct conjugacy classes of E that make up the normal subgroup $\tilde{N}$.

As can be seen from the tables, most of the groups appearing in them are semidirect products $N \times_\lambda K$, with N abelian. Assuming N, H and the action λ to be known, in the next lemma we determine Δ_G from Δ_H and the action λ.

LEMMA 2.11. *Suppose that $G = N \times_\lambda K$, with N abelian. Let $\{h_1 = 1, h_2, \dots, h_t\}$ be a complete system of representatives from distinct conjugacy classes of K. Set $\tilde{D}_i = C_K(h_i)[h_i, N]/[h_i, N]$, $\tilde{N}_i = N/[h_i, N]$ and $E_i = NC_K(h_i)/[h_i, N]$ for all $i = 1, \dots, t$. Then $E_i = \tilde{N}_i \times_\lambda \tilde{D}_i$, $\tilde{D}_i$ is isomorphic to $C_K(h_i)$, and the following propositions hold*:

(1) $r(G) = r_G(N) + r_{E_2}(\tilde{N}_2) + \cdots + r_{E_t}(\tilde{N}_t)$.

(2) *If $\{\tilde{m}_{i1} = \bar{1}, \dots, \tilde{m}_{is_i}\}$ is a complete system of representatives from distinct conjugacy classes of E_i that make up the normal subgroup $\tilde{N}_i$ ordered so that $|C_{\tilde{D}_i}(\tilde{m}_{ik})| \geqq |C_{\tilde{D}_i}(\tilde{m}_{ik+1})|$ for all $i = 2, \dots, t$ and $k = 1, \dots, s_i$, then we have*

(a) $|C_N(h_i)| = 1 + \Sigma_{k=2}^{s_i} |\tilde{D}_i : C_{\tilde{D}_i}(\tilde{m}_{ik})|$,

(b) $\Delta_{h_i} = (|C_N(h_i)| \cdot (|C_K(h_i)|/|\tilde{D}_i : C_{\tilde{D}_i}(\tilde{m}_{is_i})|), \dots,$
$|C_N(h_i)| \cdot (|C_K(h_i)|/|\tilde{D}_i : C_{\tilde{D}_i}(\tilde{m}_{is_i})|), |C_N(h_i)| \cdot |C_K(h_i)|)$,

and $\{\mathrm{Cl}_G(hm_{ij}) \mid 2 \leqq i \leqq t, 1 \leqq j \leqq s_i\}$ is the set of all conjugacy classes of G that make up the normal set $G - N$.

(3) *$E_i = \tilde{N}_i \times \tilde{D}_i$ if and only if $r_G(h_iN) = |C_N(h_i)|$. In this case we have $\Delta_{h_i} = (|C_N(h_i)|\,|C_K(h_i)|, \dots, |C_N(h_i)|\,|C_K(h_i)|)$, and in particular, if $|C_N(h_i)| = 2$, then $r_G(h_iN) = 2$ and $\Delta_{h_i} = (2|C_K(h_i)|, 2|C_K(h_i)|)$.*

(4) *The group $\tilde{D}_i$ acts transitively on $\tilde{N}_i^* = \tilde{N}_i - \{\bar{1}\}$, by conjugation, iff $r_G(h_iN) = 2$. In this case we have*

$$\Delta_{h_i} = (|C_N(h_i)| \cdot (|C_K(h_i)|/(|C_N(h_i)| - 1)), |C_N(h_i)|\,|C_K(h_i)|).$$

(5) *$r_G(h_iN) = 1$ iff h_i acts by conjugation on N^* as a fixed-point-free automorphism.*

PROOF. The propositions (1) and (2) are an immediate consequence of Lemmas 2.1 and 2.10, because $C_K(h_i)N/N = C_{\bar{K}}(\bar{h}_i)$ for all $i = 2, \dots, t$.

On the other hand, (3), (4) and (5) are very interesting cases that follow directly from (2).

EXAMPLE. We consider the unique non-split extension of C_2^4 by A_6,

$$C_2^4 \times_\lambda A_6 = (\langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle) \times_\lambda \langle a, b \mid a^5 = b^5 = (ab)^2 = (a^{-1}b)^4 = 1 \rangle,$$

with relations $x_1^a = x_4$, $x_2^a = x_1x_4$, $x_3^a = x_2x_4$, $x_4^a = x_3x_4$, $x_1^b = x_1x_2$, $x_2^b = x_2x_3$, $x_3^b = x_2x_4$, $x_4^b = x_1x_3x_4$. In this group, a (respect. b) corresponds to the permutation (15432) (respect. (13456)).

We have $\Delta_{A_6} = (360,9,9,8,5,5,4) = (|C_{A_6}(h_1)|,\dots,|C_{A_6}(h_7)|)$ and if the tuple $(h_1,h_2,\dots,h_7)$ corresponds to the tuple

$$(1,(123),(123)(456),(12)(34),(12345),(12354),(1234)),$$

then one sees that h_3, h_5 and h_6 act f.p.f. on N^*, hence $r_G(h_iN) = 1$ for $i = 3,5,6$ and $\Delta_{h_3} = (9)$, $\Delta_{h_5} = (5)$, $\Delta_{h_6} = (5)$. Furthermore, with the notation from the above lemma, we have:

(1) $\tilde{N}_2 \simeq C_2^2$, $\tilde{D}_2 \simeq C_3^2$ and $E_2 \simeq C_2^2 \times_\lambda C_3^2 = C_3 \times A_4$, hence $\tilde{D}_2$ acts transitively on $\tilde{N}_2^*$, so $r_G(h_2N) = 2$ and $\Delta_{h_2} = (4\cdot 9/3, 4\cdot 9) = (12,36)$;

(2) $\tilde{N}_6 \simeq C_2^2$, $\tilde{D}_6 \simeq D_8$ and $E_6 = C_2^2 \times_\lambda D_8 = \langle w_1, w_2\rangle\langle c,d \mid c^4 = d^2 = cc^d = 1\rangle$ with relations $w_1^c = w_1$, $w_2^c = w_1w_2$, $w_1^d = w_1$, $w_2^d = w_2$, hence $r_G(h_7N) = 3$ and $\Delta_{h_7} = (4\cdot 8/2, 4\cdot 8/1, 4\cdot 8) = (16,32,32)$, because the conjugacy classes of $\tilde{N}_7$ have cardinality 1,1 and 2 respectively. Thus $r(G) = 12$ and

$$\Delta_G = (5760,384,36,32,32,16,12,9,8,8,5,5).$$

Lemma 2.12. *Let G be a simple group satisfying $|\{o(g) \mid g \in G\}| = 4$ or 5. Then G is isomorphic to one of the following groups*: A_5, PSL(2,7), PSL(2,9), SL(2,8).

Proof. Let P be a 2-Sylow subgroup of G and p,q two odd primes dividing the order of G. Set $o(g) = 2$. If $\exp P = 2$, then $|C_G(g)|$ is divisible by at most two prime numbers, hence $C_G(g)$ is a solvable group, so G is isomorphic to PSL$(2,s)$ with either $s > 3$ and $s \equiv 3,5 \pmod 8$ or $s = 2^m$ (cf. [7] p. 484), and necessarily either $G \simeq$ PSL(2,5) or $G \simeq$ SL(2,8), because SL$(2,s)$ has elements of orders $s-1$ and $s+1$. On the other hand, if $\exp P \neq 2$, then $\{o(g) \mid g \in G\} = \{1,2,4,p,q\}$, hence G is a C-group. Thus, either $G \simeq$ PSL(2,7) or $G \simeq$ PSL(2,9) (cf. [10] p. 465).

Lemma 2.13. *Let ϕ be an automorphism of G and p an odd prime. We have*:

(i) *If $o(\phi) = p$ and $|C_G(\phi)| = 1$, then G is a nilpotent group.*

(ii) *If $o(\phi) = p$ and $|C_G(\phi)| = 2$, then G is a solvable group.*

(iii) *If $o(\phi) = 4$ and $|C_G(\phi)| = 1$, then G is a solvable group.*

Proof. (i) cf. [7] p. 337.

(ii) cf. [5].

(iii) cf. [7] p. 341.

LEMMA 2.14. *Let G be a Frobenius group with kernel K and complement D. Then:*

(i) $|D|$ *is a divisor of* $|K|-1$.

(ii) $r(G)=r(D)+(r(K)-1)/|D|$.

(iii) *K is a nilpotent group.*

(iv) *If 2 is a divisor of $|D|$ then K is an abelian group.*

(v) *The Sylow 2-subgroups of G are either cyclic groups or quaternion groups.*

PROOF. cf. [7] p. 339.

LEMMA 2.15. *Let G be a group with $S(G)$ abelian. Suppose that there exists $\bar{H} \leqq \bar{G} = G/S(G)$ such that $\bar{H} \simeq C_p \times C_p$, p a prime, and $\bar{H}-\{\bar{1}\}$ is a set of conjugate elements of $\bar{G}$. Then $\bar{H}$ does not act* f.p.f. *on $O_q(S(G))$ for each prime number $q \neq p$ and*

$$|C_G(h_1)\cap S(G)| = |C_G(h_2)\cap S(G)| \qquad \text{for any } h_1, h_2 \in H - S(G).$$

Further, if $p=2$ and $|C_G(h)\cap O_q(S(G))| = q^t$, q an odd prime, then $|O_q(S(G))|$ is a divisor of q^{3t}.

PROOF. Let $q \neq p$ be a prime divisor of $|S(G)|$. (2.14) yields that $\bar{H}$ does not act f.p.f. on $O_q(S(G))$. Moreover, if $h_1, h_2 \in H-\{1\}$, then $C_G(h_1)\cap S(G)$ is conjugate to $C_G(h_2)\cap S(G)$, hence $|C_G(h_1)\cap S(G)| = |C_G(h_2)\cap S(G)|$. Finally, if $\bar{H} = \langle \bar{h}_1\rangle \times \langle \bar{h}_2\rangle \simeq C_2^2$ and $L_i = C_G(h_i)\cap O_q(S(G))$, then $L_i \leq E = O_q(S(G))\langle h_1, h_2\rangle$, hence (2.8) implies that there exists $D \trianglelefteq E$ such that $O_q(S(G)) = L_1L_2 \times D$; so, $d^{h_1} = d^{-1} = d^{h_2}$ for each $d \in D$. Consequently D is a subgroup of $C_G(h_1h_2)\cap O_q(S(G))$ and $|O_q(S(G))|$ is a divisor of q^{3t}.

LEMMA 2.16. *Let G be a group with $S(G)$ abelian. Set $\bar{G} = G/S(G)$, $\bar{b}$ an element of order 2 of $\bar{G}$, and $M \trianglelefteq G$ such that $M \leqq S(G)$. If $C_G(b)\cap M = 1$, then $z^y = z$ for any $z \in M$ and $y \in G$ such that $\overline{by}$ is a conjugate element to $\bar{b}$ in $\bar{G}$.*

PROOF. If $\overline{by} = \bar{b}^{\bar{g}}$ for some $\bar{g} \in \bar{G}$, then $C_G(by)\cap M = (C_G(b)\cap M)^g = 1$, hence $z^b = z^{-1} = z^{by}$ for each $z \in M$. Therefore $z^y = z$.

Let Γ be the class of the nilpotent finite groups. The families

$$\Psi_j = \Phi_j \cap \Gamma, \qquad 1 \leqq j \leqq 15$$

are classified in [22]. We have:

THEOREM 2.17. (i) $\Psi_1 = \{F_{t,1} \mid t \in N\}$.

(ii) $\Psi_2 = \{C_3\}$.

(iii) $\Psi_3 = \{C_4\}$.
(iv) $\Psi_4 = \{C_5, C_6, D_8, Q_8\}$.
(v) $\Psi_5 = \{C_2 \times C_4, C_3 \times C_3\}$.
(vi) $\Psi_6 = \{C_7, Q_{16}, D_{16}, \mathrm{SD}_{16}\}$.
(vii) $\Psi_7 = \{C_8, C_2 \times D_8, C_2 \times Q_8, C_4 \times_\lambda C_4, (C_4 \times C_2) \times_{\lambda_1} C_2\}$, *where* $C_4 \times_\lambda C_4 = \langle a \rangle \times_\lambda \langle b \rangle$ *with* λ *given by* $a^b = a^{-1}$ *and* $(C_4 \times C_2) \times_{\lambda_1} C_2 = (\langle a \rangle \times \langle b \rangle) \times_{\lambda_1} \langle c \rangle$ *with* λ_1 *given by* $a^c = ab$, $b^c = b$.
(viii) $\Psi_8 = \{C_9, C_6 \times C_2, C_{10}\}$.
(ix) $\Psi_9 = \{C_4 \times C_2 \times C_2, C_8 \times_\lambda C_2, (C_4 \times C_2) \times_{\lambda_2} C_2\}$, *where* $C_8 \times_\lambda C_2 = \langle a \rangle \times_\lambda \langle b \rangle$ *with* λ *given by* $a^b = a^5$ *and* $(C_4 \times C_2) \times_{\lambda_2} C_2$ *with* λ_2 *given by* $a^c = a$, $b^c = a^2 b$.
(x) $\Psi_{10} = \{C_{11}, C_{12}, Q_1, Q_2, D_{32}, Q_{32}, \mathrm{SD}_{32}\} \cup 2^5\Gamma_6 \cup 2^5\Gamma_7$.

Now, the finite groups G satisfying the conditions $\alpha(G) = 1,2$ or 3 are described in the next three lemmas (cf. [23]).

LEMMA 2.18. *G is a finite group satisfying $\alpha(G) = 1$ if and only if $G = H \times_f C_2$. This group satisfies $r(G) = 2 + ((|H| - 1)/2)$.*

LEMMA 2.19. *G is a finite group satisfying $\alpha(G) = 2$ if and only if G is isomorphic to one of the following groups*: (i) $Y \times_f C_3$, (ii) $C_2 \times (H \times_f C_2)$, (iii) $H \times_\lambda C_4 = H \times_\lambda \langle b \rangle$ *with relations* $h^b = h^{-1}$ *for each* $h \in H$, (iv) C_4. *Furthermore* $r(G) = 3 + |H|$ *for all the groups which appear in* (ii) *or* (iii).

LEMMA 2.20. *G is a finite group satisfying $\alpha(G) = 3$ if and only if G is one of the following groups*:
(i) $C_3 \times (H \times_f C_2)$,
(ii) $H \times_f C_4$,
(iii) $(A_5 \times H) \times_\lambda C_2 = (A_5 \times H) \times_\lambda \langle b \rangle$, *where* λ *is given by* $A_5\langle b \rangle \simeq \Sigma_5$ *and* $h^b = h^{-1}$ *for each* $h \in H$. *This group satisfies* $r = 6 + (5|H| - 3)/2$.
(iv) $(A_6 \times H) \cdot C_4 = (A_6 \times H) \cdot \langle b \rangle$ *with* $A_6\langle b \rangle \simeq \mathrm{PGL}^*(2,9) = M_9$ *and* $h^b = h^{-1}$ *for every* $h \in H$. *This group satisfies* $r = 6 + (7|H| - 3)/2$.
(v) $M_9, \Sigma_5, \Sigma_4, D_8$ *or* Q_8.

REMARKS. (1) If N is a normal subgroup of G, then $\alpha(G/N) \leqq \alpha(G)$. It is an immediate consequence of the fact that $S(G)N/N \leqq S(G/N)$.

(2) $\alpha(\mathrm{PGL}(2,q)) = (q+1)/2$, if q is an odd number greater than 3.

3. Groups with $S(G)$ non-solvable and $\alpha(G) \leqq 9$

LEMMA 3.1. *Let $G \in \Phi_t$ be such that $S(G)$ is non-solvable. Then $t \geqq 3\beta(G) + \alpha(G)$.*

PROOF. Set $\{L_1,\dots,L_{\beta(G)}\}$ the set of minimal normal subgroups of G. As $S(G)$ is non-solvable, we can suppose that L_1 is non-solvable.

Clearly we have $(L_1\times L_i)\cap(L_1\times L_j)=L_1$ for every $i\neq j\geqq 2$ and $r_G(L_1)\geqq 3$, because $|L_1|$ is divisible by at least three prime numbers. Set $T=L_1\cup\cdots\cup L_{\beta(G)}$, then $(L_1\times L_i)\cap T=L_1\cup L_i$, hence $L_1\times L_i-(L_1\cup L_i)$ is a subset of $S(G)-T$. Thus $r_G(S(G)-T)\geqq 3(\beta(G)-1)$. Moreover $r_G(G-S(G))=\alpha(G)$ and $r_G(T)\geqq 1+(\beta(G)-1)+3$, hence

$$r(G)\geqq \alpha(G)+(\beta(G)-1)3+\beta(G)+3=4\beta(G)+\alpha(G).$$

THEOREM 3.2. *Let G be a finite group such that $S(G)$ is non-solvable and $\beta(G)=r(G)-j$ with $1\leqq j\leqq 10$. Then G is isomorphic to one of the following groups*: $A_5, A_6, A_7, \Sigma_5, \Sigma_6, A_5\times C_2$, PSL(2,7), PGL(2,7), PSL(2,7)$\times C_2$, PGL(2,9), PGL*(2,9), SL(2,8), PΓL(2,8), PSL(2,11), PSL(2,13), PSL(2,17), PSL(3,4), M_{11}, Sz(8), $(A_5\times C_3)\times_\lambda C_2$ with $A_5C_2\simeq\Sigma_5$ *and* $C_3C_2\simeq\Sigma_3$.

PROOF. If $S(G)=G$, G is completely reducible, hence $G=G_1\times\cdots\times G_s\times Z(G)$ with the $G_1,\dots,G_s$ non-abelian simple groups. Then

$$r(G)=r(G_1)\cdots r(G_s)\cdot|Z(G)|\geqq 5^s\cdot|Z(G)| \quad\text{and}\quad \beta(G)\leqq s+|Z(G)|-1.$$

So $5^s\cdot|Z(G)|-(s+|Z(G)|-1)\leqq r(G)-\beta(G)=a\leqq 10$ forces $s=1$ and $|Z(G)|\leqq 2$. Consequently, either $G\simeq A_5\times C_2$, or $G\simeq$ PSL(2,7)$\times C_2$, or G is a simple group with $r(G)\leqq 11$, hence from [1], G is isomorphic to one of the following groups: A_5, PSL(2,7), A_6, PSL(2,11), A_7, PSL(2,13), SL(2,8), PSL(3,4), M_{11}, Sz(8), PSL(2,17).

Thus we can suppose $S(G)<G$, i.e. $\alpha(G)\geqq 1$. Further, (2.18) and (2.19) imply $\alpha(G)\geqq 3$. If $\alpha(G)=3$, then G is isomorphic to one of the following groups: M_9, Σ_5, or $(A_5\times C_3)\times_\lambda C_2$, by (2.20). Now, we suppose $\alpha(G)\geqq 4$, (3.1) yields $3\beta(G)+\alpha(G)\leqq 10$, hence $\beta(G)\leqq 2$ and necessarily $\beta(G)=1$. If $S(G)=A\times A$ with A a non-abelian simple group, then $\alpha(G)=4$, $r(G)\leqq 12$ and $r(G/S(G))\leqq 5$.

If $r(G/S(G))=5=\alpha(G)+1$, then (2.3) yields $|C_G(x)|=|C_{\bar G}(\bar x)|$ for each $x\in G-S(G)$ and (2.13) (or (2.3)(ii)) implies that $S(G)$ is solvable (this is deduced from an inspection of the tuples $\Delta_{\bar G}$ for $r(\bar G)=5$), which is impossible. Thus $r(G/S(G))\leqq 4$.

If $G/S(G)\simeq C_p$, with p prime, then $r(G)=ps+(r(A)^2-s)/p\geqq 13$, impossible. So, we can suppose that $G/S(G)$ is isomorphic to one of the following groups: Σ_3, C_4, $C_2\times C_2$, A_4, D_{10}.

If $G/S(G)\simeq\Sigma_3$, (2.1) implies $4=r_G(aS(G))+r_G(bS(G))$ with $o(\bar a)=3$ and

$o(\bar{b})=2$. On the other hand, (2.13) implies $r_G(aS(G))\geqq 3$ and $r_G(bS(G))\geqq 2$; this is impossible.

If $G/S(G)\simeq C_4$, (2.1) implies $4=r_G(bS(G))+r_G(b^{-1}S(G))+r_G(b^2S(G))$ with $o(\bar{b})=4$, hence necessarily $r_G(bS(G))=1$, $C_G(b)=\langle b\rangle$ and $S(G)$ is solvable by (2.13)(iii), impossible.

If $G/S(G)\simeq C_2\times C_2$, then (2.1)(i) implies that there exists $b\in G-S(G)$ such that $r_G(bS(G))=1$, hence $|C_G(b)|=4$. Let P be a Sylow 2-subgroup of G and $g\in G$ such that $P/\langle g\rangle\simeq C_2$, then $g^2\in S(G)$ and $S(G)$ have cyclic Sylow 2-subgroups, impossible.

If $G/S(G)\simeq A_4$, we have $4=r_G(aS(G))+r_G(bS(G))+r_G(b^{-1}S(G))$ with $o(\bar{a})=2$ and $o(\bar{b})=3$, hence necessarily $r_G(bS(G))=1$ and $S(G)$ is solvable by (2.13)(i). Similarly the case $G/S(G)\simeq D_{10}$ cannot arise here. Thus we can suppose $\beta(G)=1$, $r(G)\leqq 11$, $S(G)$ is a non-abelian simple group and $S(G)\leqq G\leqq \mathrm{Aut}(S(G))$. Also, arguing as above, we can suppose $\alpha(G)\geqq 5$ (the case $G/S(G)\simeq C_3$ is left out by (2.13), if $\alpha(G)=4$) or $\alpha(G)=4$ and $G/S(G)\simeq C_2$.

If $\alpha(G)\geqq 6$, then $r_G(S(G))\leqq 5$, hence $|\{o(g)\mid g\in S(G)\}|=4$ or 5 and from (2.12) we obtain that $S(G)$ is isomorphic to one of the following groups: A_5, $\mathrm{PSL}(2,7)$, A_6, $\mathrm{SL}(2,8)$, hence necessarily G is isomorphic to $\mathrm{Aut}(\mathrm{SL}(2,8))=\mathrm{P\Gamma L}(2,8)$ ($\alpha(\mathrm{P\Gamma L}(2,8))=6$).

Now we suppose that $\alpha(G)=5$. Then $r(G/S(G))\leqq 6$.

If $r(G/S(G))=6=\alpha(G)+1$, then $|C_G(x)|=|C_{\bar{G}}(\bar{x})|$ for each $x\in G-S(G)$ and looking at the possible values of the components of the 6-tuples $\Delta_{\bar{G}}$, we deduce from (2.13) (or (2.3)(ii)) that $S(G)$ is solvable, which is impossible.

If $r(G/S(G))=4$ or 5, then there exists $x\in G-S(G)$ such that $r_G(xS(G))=1$, by (2.1)(i), so $|C_G(x)|=|C_{\bar{G}}(\bar{x})|$ and

$$|C_G(x)|\in\{2,3,4,5,7,8\}.$$

If $|C_G(x)|$ is a prime number, then (2.13) implies that $S(G)$ is solvable, impossible.

If $|C_G(x)|=4$, (2.13)(iii) implies $o(\bar{x})=2$, and one observation of the tuples $\Delta_{\bar{G}}$ implies that $\bar{G}$ has Sylow 2-subgroups isomorphic to $C_2\times C_2$. Let P be a Sylow 2-subgroup of G and $g\in P$ such that $P/\langle g\rangle\simeq C_2$, then $g^2\in S(G)$ and $S(G)$ is solvable, impossible.

If $|C_G(x)|=8$, then $G/S(G)$ is isomorphic to D_8, Q_8, or Σ_4, and there exists $\bar{y}\in\bar{G}$ with $r_G(yS(G))=1$ which does not conjugate with $\bar{x}$, hence $|C_G(y)|=4$ or 3 and, reasoning as before, $S(G)$ is solvable, impossible.

Suppose $G/S(G)=\langle\bar{b}\rangle\simeq C_3$, then $5=r_G(bS(G))+r_G(b^{-1}S(G))$, but $r_G(bS(G))\geqq 3$ and $r_G(b^{-1}S(G))\geqq 3$, by (2.13), impossible.

Suppose $G/S(G) \simeq \Sigma_3$, then $5 = r_G(aS(G)) + r_G(bS(G))$ with $o(\bar{a}) = 3$ and $o(\bar{b}) = 2$. (2.13) yields $r_G(aS(G)) \geqq 3$, hence $r_G(bS(G)) = 2$, $|C_G(b)| = 4$ and $S(G)$ has cyclic Sylow 2-subgroups, this is impossible.

Finally, we consider only the case $G/S(G) \simeq C_2$ and $4 \leqq \alpha(G) \leqq 5$. We have $r_G(S(G)) \leqq 7$, hence $r(S(G)) \leqq 2.5 + 2 = 12$, because $C_G(a) \cap S(G) \neq 1$, hence [1] implies that $S(G)$ is isomorphic to one of the following groups: A_5, PSL(2,7), A_6, PSL(2,11), A_7, PSL(2,8), PSL(2,13), PSL(3,4), M_{11}, Sz(8), PSL(2,17), M_{22}, PSL(3,3), or PSL(2,19). By considering the structure of their automorphism groups we obtain the following groups: PGL(2,7) (α(PGL(2,7)) = 4), Σ_6 ($\alpha(\Sigma_6) = 5$), or PGL(2,9) (α(PGL(2,9)) = 5).

4. Non-nilpotent groups with $S(G)$ solvable. The families Φ_j

LEMMA 4.1. *Let G be a non-nilpotent group with $S(G)$ solvable and satisfying $\alpha(G) = 4$. Then G is isomorphic to one of the following groups*: (i) $(C_2 \times C_2) \times (H \times_f C_2)$, (ii) $H \times_f Q_8$, (iii) $C_2 \times (Y \times_f C_3)$, (iv) $Y \times_f C_5$, (v) $C_9 \times_f C_2$, (vi) $C_2 \times (H \times_\lambda C_4) = C_2 \times (H \times_\lambda \langle b \rangle)$ *with relations $h^b = h^{-1}$ for each $h \in H$; this group satisfies $r = 6 + 2|H|$*, (vii) $(C_3 \times H) \times_\lambda C_4 = (\langle x \rangle \times H) \times_\lambda \langle b \rangle$ *with relations $x^b = x^{-1}$, $H\langle b \rangle = H \times_f \langle b \rangle$; this group satisfies $r(G) = 6 + 3(|H| - 1)/4$.*

PROOF. cf. [23].

LEMMA 4.2. *Let G be a non-nilpotent group with $S(G)$ abelian and satisfying the conditions $5 \leqq \alpha(G) \leqq 9$ and $r(G/S(G)) \leqq 6$. Then G is isomorphic to one group of Table 5.*

REMARK. In Table 5, $a_H = |H| - 1$.

PROOF. Since $r(G/S(G)) \leqq 6$, $G/S(G)$ is isomorphic to one of the groups in Table 1 with at most 6 conjugate classes.

(1) Suppose $G/S(G) = \langle \bar{b} \rangle \simeq C_2$. Then $\alpha(G) = r_G(bS(G)) = |C_G(b) \cap S(G)|$. Set $L = C_G(b) \cap S(G)$, L is a normal subgroup of G and (2.7) implies that there exists $D \trianglelefteq G$ such that $S(G) = L \times D$ and $C_D(b) = 1$. Now, fixing the values of $\alpha(G)$, we obtain the desired groups.

(2) Suppose $G/S(G) = \langle \bar{a} \rangle \times_f \langle \bar{b} \rangle \simeq C_p \times_f C_q$, where p and q are prime numbers such that q divides $p - 1$, then

$$(8) \qquad \alpha(G) = ((p-1)/q) \cdot |C_G(a) \cap S(G)| + (q-1)|C_G(b) \cap S(G)|.$$

Set $L = C_G(a) \cap S(G) \trianglelefteq G$ and $D \trianglelefteq G$ such that $S(G) = L \times D$, then $\langle a \rangle$ acts f.p.f. on D. Now, one simple study of the solutions from (8) for $5 \leqq \alpha(G) \leqq 9$

TABLE 5

$G/S(G)$	$\alpha(G)$	G	$r(G)$
	5	$(H \times_f C_2) \times C_5$	$10 + (5 \cdot a_H \cdot 2^{-1})$
	6	$(H \times_f C_2) \times C_6$	$12 + (3 \cdot a_H)$
	6	$C_3 \times (H \times_\lambda C_4) = C_3 \times (H \times_\lambda \langle b \rangle),\ h^b = h^{-1}\ \forall h \in H$	$12 + (3 \cdot a_H)$
C_2	7	$C_7 \times (H \times_f C_2)$	$14 + (7 \cdot a_H \cdot 2^{-1})$
	8	$(C_2 \times C_2 \times C_2) \times (H \times_f C_2)$	$16 + (4 \cdot a_H)$
	8	$C_2^2 \times (H \times_\lambda C_4) = C_2^2 \times (H \times_\lambda \langle b \rangle),\ h^b = h^{-1}\ \forall h \in H$	$16 + (4 \cdot a_H)$
	9	$C_3^2 \times (H \times_f C_2)$	$18 + (9 \cdot a_H \cdot 2^{-1})$
	6	$C_3 \times (Y \times_f C_3)$	$8 + \lvert Y \rvert$
C_3	6	$Y \times_\lambda C_9 = Y \times_\lambda \langle b \rangle$ with $y^{b^3} = y\ C_y(b) = 1$	$8 + \lvert Y \rvert$
	8	$(C_2 \times C_2) \times (Y \times_f C_3)$	$12 + (4 \cdot a_Y \cdot 3^{-1})$
	5	$(C_2 \times C_2) \times_\lambda (C_9 \times_f C_2) = (\langle x \rangle \times \langle y \rangle) \times_\lambda (\langle a \rangle \times_f \langle b \rangle)$	9
		$x^a = y,\ y^a = xy,\ x^b = x,\ y^b = xy$	
	5	$(C_3 \times C_2 \times C_2) \times_\lambda \Sigma_3 = (\langle z \rangle \times \langle x \rangle \times \langle y \rangle) \times_\lambda \langle a, b \rangle$	9
		$z^a = z,\ z^b = z^{-1},\ x^a = y,\ y^a = xy,\ x^b = x,\ y^b = xy$	
	5	$C_2^4 \times_\lambda \Sigma_3 = (\langle x_1 \rangle \times \langle y_1 \rangle \times \langle x_2 \rangle \times \langle y_2 \rangle) \times_\lambda \langle a, b \rangle$	10
		$x_i^a = y_i,\ y_i^a = x_i y_i,\ x_i^b = x_i,\ y_i^b = x_i y_i,\ i = 1,2$	
	6	$(C_5 \times C_5) \times_\lambda \Sigma_3 = (\langle x \rangle \times \langle y \rangle) \times_\lambda \langle a, b \rangle$	13
		$x^a = y,\ y^a = x^{-1} y^{-1},\ x^b = x,\ y^b = x^{-1} y^{-1}$	
	6	$(C_2 \times C_2) \times_\lambda DC_3 = (\langle x \rangle \times \langle y \rangle) \times_\lambda \langle a, b \rangle$	10
		$x^a = y,\ y^a = xy,\ x^b = x,\ y^b = xy$	
	6	$C_2 \times \Sigma_4$	10
Σ_3	7	$C_2^2 \times_\lambda (C_{15} \times_f C_2) = (\langle x \rangle \times \langle y \rangle) \times_\lambda \langle a, b \rangle$	13
		$x^a = y,\ y^a = xy,\ x^b = x,\ y^b = xy$	
	7	$C_2^4 \times_\lambda (C_9 \times_f C_2) = (\langle x_1 \rangle \times \langle y_1 \rangle \times \langle x_2 \rangle \times \langle y_2 \rangle) \times_\lambda \langle a, b \rangle$	18
		$x_i^a = y_i,\ y_i^a = x_i y_i = y_i^b,\ x_i^b = x_1,\ i = 1,2$	
	7	$C_2^4 \times_\lambda ((C_3 \times C_3) \times_f C_2) = (\Pi \langle x_i \rangle \times \langle y_i \rangle) \times_\lambda \langle a_1, a_2, b \rangle$	18
		$x_i^{a_1} = y_i = y_i^{a_2},\ y_i^{a_1} = x_i y_i = y_i^b,\ x_i^b = x_i = x_i^{a_2},\ i = 1,2$	
	8	$C_9 \times_\lambda C_4 = \langle a \rangle \times_\lambda \langle b \rangle,\ a^b = a^{-1}$	12
	8	$C_2^4 \times_\lambda \Sigma_3 = (\langle x_1 \rangle \times \langle y_1 \rangle \times \langle x_2 \rangle \times \langle y_2 \rangle) \times_\lambda \langle a, b \rangle\ x_1^a = x_1 = x_1^b$	14
		$x_2^a = y_2,\ y_1^a = y_1,\ y_2^a = x_2 y_2 = y_2^b,\ x_2^b = x_2,\ y_1^b = x_1 y_1,\ x_2^b = x_2$	
	8	$C_2 \times (C_9 \times_f C_2)$	12
	8	$(C_5 \times C_5) \times_\lambda ((C_3 \times C_3) \times_f C_2) = (\langle x, y \rangle) \times_\lambda \langle a_1, a_2, b \rangle$	24
		$x^{a_1} = x = x^b,\ y^{a_1} = y = x^{a_2},\ y^{a_2} = x^{-1} y^{-1} = y^b$	
	8	$(C_5 \times C_5) \times_\lambda (C_9 \times_f C_2) = \langle x, y \rangle \times_\lambda \langle a, b \rangle$	24
		$x^a = y,\ y^a = x^{-1} y^{-1} = y^b,\ x^b = x$	
	8	$(C_7 \times C_7) \times_\lambda \Sigma_3 = \langle x, y \rangle \times_\lambda \langle a, b \rangle$	20
		$x^a = y,\ y^a = x^{-1} y^{-1} = y^b,\ x^b = x$	
Σ_3	9	$C_3 \times \Sigma_4$	15
	9	$C_2^6 \times_\lambda \Sigma_3 = (\Pi \langle x_i \rangle \times \langle y_i \rangle) \times_\lambda \langle a, b \rangle$	24
		$x_i^a = y_i,\ y_i^a = x_i y_i,\ x_i^b = x_i,\ y_i^b = x_i y_i,\ i = 1,2,3$	
	9	$C_2^4 \times_\lambda (C_{15} \times_f C_2) = (\Pi \langle x_i \rangle \times \langle y_i \rangle) \times_\lambda \langle a, b \rangle$	26
		$x_i^a = y_i,\ y_i^a = x_i y_i,\ x_i^b = x_i,\ y_i^b = x_i y_i,\ i = 1,2$	
	9	$(C_2 \times C_2) \times_\lambda (C_{21} \times_f C_2) = \langle x, y \rangle \times_\lambda \langle a, b \rangle$	17
		$x^a = y,\ y^a = xy,\ x^b = x,\ y^b = xy$	
	5	$\Sigma_3 \times \Sigma_3$	9
	5	$C_{12} \times_\lambda C_2 = \langle a \rangle \times_\lambda \langle b \rangle \quad a^b = a^{-1}$	9

TABLE 5 (contd.)

$G/S(G)$	$\alpha(G)$	G	$r(G)$
	5	$C_3 \times_\lambda D_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b,\ a^b = a^{-1}$	9
	5	$C_3 \times_\lambda Q_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b$	9
	6	$\Sigma_3 \times D_{10}$	12
	6	$C_3^3 \times_\lambda C_2^2 = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle$ $x^a = x,\ y^a = y^{-1},\ z^a = z^{-1},\ x^b = x^{-1},\ y^b = y,\ z^b = z^{-1}$	15
	7	$\Sigma_3 \times D_{14}$	15
	7	$D_{10} \times D_{10}$	16
	7	$C_5 \times_\lambda D_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b$	13
C_2^2	7	$C_5 \times_\lambda Q_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b$	13
	7	$C_{20} \times_\lambda C_2 = \langle a \rangle \times_\lambda \langle b \rangle \quad a^b = a^{-1}$	13
	7	$(C_3 \times C_3) \times_\lambda D_8 = \langle x, y \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b,$ $y^a = y,\ y^b = y^{-1},\ a^b = a^{-1}$	15
	7	$(C_3 \times C_3) \times_\lambda D_8 = \langle x, y \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b,$ $y^a = y^{-1},\ y^b = y$	15
	7	$(C_3 \times C_3) \times_\lambda Q_8 = \langle x, y \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1},\ x^b = x,$ $y^a = y^{-1} = y^b$	15
	7	$(C_3^2 \times C_5) \times_\lambda (C_2^2) = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^a = x,\ x^b = x^{-1},$ $y^a = y^{-1},\ y^b = y,\ z^a = z^{-1} = z^b,\ x^3 = y^3 = z^5 = 1.$	21
	8	$C_3^3 \times_\lambda C_2^2 = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1},\ x^b = x,$ $y^a = y^{-1} = y^b,\ z^a = z^{-1} = z^b$	18
	8	$D_{10} \times D_{14}$	20
	8	$(C_3 \times C_3 \times C_7) \times_\lambda (C_2 \times C_2) = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^3 = y^3 = 1$ $z^7 = 1,\ z^a = z^{-1} = z^b,\ y^a = y^{-1},\ y^b = y,\ x^a = x,\ x^b = x^{-1}$	27
	8	$(C_3 \times C_5 \times C_5) \times_\lambda (C_2 \times C_2) = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^3 = y^5 = 1$ $z^5 = 1,\ z^a = z^{-1} = z^b,\ y^a = y^{-1},\ y^b = y,\ x^a = x,\ x^b = x^{-1}$	30
	8	$C_4 \times \Sigma_3$	12
	9	$\Sigma_3 \times D_{22}$	21
	9	$D_{10} \times ((C_3 \times C_3) \times_f C_2)$	24
	9	$D_{14} \times D_{14}$	25
	9	$C_7 \times_\lambda D_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad a^b = a^{-1},\ x^a = x^{-1} = x^b$	17
	9	$C_{28} \times_\lambda C_2 = \langle a \rangle \times_\lambda \langle b \rangle \quad a^b = a^{-1}$	17
	9	$C_7 \times_\lambda Q_8 = \langle x \rangle \times_\lambda \langle a, b \rangle \quad x^a = x^{-1} = x^b,$	17
	9	$(C_3 \times C_5) \times_{\lambda_1} D_8 = \langle x, y \rangle \times_{\lambda_1} \langle a, b \rangle \quad x^3 = y^5 = 1,$ $x^a = x,\ y^a = y^{-1} = y^b,\ x^b = x^{-1}$	21
	9	$(C_3 \times C_5) \times_{\lambda_2} D_8 = \langle x, y \rangle \times_{\lambda_2} \langle a, b \rangle \quad x^3 = y^5 = 1,$ $x^a = x^{-1} = x^b,\ y^a = y,\ y^b = y^{-1}$	21
	9	$(C_3 \times C_5) \times_{\lambda_3} D_8 = \langle x, y \rangle \times_{\lambda_3} \langle a, b \rangle \quad x^3 = y^5 = 1,$ $x^a = x^{-1},\ x^b = x,\ y^a = y^{-1} = y^b$	21
C_2^2	9	$(C_3 \times C_5) \times_\lambda Q_8 = \langle x, y \rangle \times_\lambda \langle a, b \rangle \quad y^a = y^{-1} = y^b,$ $x^a = x^{-1},\ x^b = x$	21
	9	$C_3^4 \times_\lambda C_2^2 = \langle x, y, z_1, z_2 \rangle \times_\lambda \langle a, b \rangle \quad x^a = x,\ x^b = x^{-1},$ $y^a = y^{-1},\ y^b = y,\ z_i^a = z_i^{-1} = z_i^b,\ i = 1,2$	33
	9	$C_5^3 \times_\lambda C_2^2 = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^a = x,\ x^b = x^{-1},$ $y^a = y^{-1},\ y^b = y,\ z^a = z^{-1} = z^b$	44
	9	$(C_3 \times C_5 \times C_7) \times_\lambda (C_2 \times C_2) = \langle x, y, z \rangle \times_\lambda \langle a, b \rangle \quad x^a = x,$ $x^b = x^{-1},\ y^a = y^{-1},\ y^b = y,\ z^a = z^{-1} = z^b$	39
	9	$C_3 \times D_8$	15
	9	$C_3 \times Q_8$	15

TABLE 5 (contd.)

$G/S(G)$	$\alpha(G)$	G	$r(G)$
	9	$(C_3\times C_3\times C_3)\times_\lambda D_8=\langle x,y,z\rangle\times_\lambda\langle a,b\rangle$ $x^a=x,\ x^b=x^{-1}$ $y^a=y^{-1},\ y^b=y,\ z^a=z^{-1}=z^b$	27
	9	$(C_3\times C_3\times C_3)\times_\lambda Q_8=\langle x,y,z\rangle\times_\lambda\langle a,b\rangle$ $x^a=x,\ x^b=x^{-1}$ $y^a=y^{-1},\ y^b=y,\ z^a=z^{-1}=z^b$	27
C_4	5	$(C_5\times H)\times_\lambda C_4=(\langle a\rangle\times H)\times_\lambda\langle b\rangle$ $a^b=a^{-1}$, $H\langle b\rangle=H\times_f\langle b\rangle$	$8+(5\cdot a_H\cdot 4^{-1})$
	6	$C_2\times(H\times_f C_4)$	$8+(a_H\cdot 2^{-1})$
	6	$H\times_\lambda C_8=H\times_\lambda\langle b\rangle$ $h^{b^4}=h\ \forall h\in H,\ C_H(b^2)=1$	$8+(a_H\cdot 2^{-1})$
	6	$(C_7\times H)\times_\lambda C_4=(\langle a\rangle\times H)\times_\lambda\langle b\rangle$ $a^b=a^{-1}$ $H\langle b\rangle=H\times_f\langle b\rangle$	$10+(7\cdot a_H\cdot 4^{-1})$
	7	$(C_3\times C_3\times H)\times_\lambda C_4=(\langle x,y\rangle\times H)\times_\lambda\langle b\rangle$ $x^b=x^{-1},\ y^b=y^{-1},\ H\langle b\rangle=H\times_f\langle b\rangle$	$12+(9\cdot a_H\cdot 4^{-1})$
	8	$(C_{11}\times H)\times_\lambda C_4=(\langle a\rangle\times H)\times_\lambda\langle b\rangle$ $a^b=a^{-1}$, $H\langle b\rangle=H\times_f\langle b\rangle$	$14+(11\cdot a_H\cdot 4^{-1})$
C_4	8	$C_2\times((C_3\times H)\times_\lambda C_4)=C_2\times((\langle a\rangle\times H)\times_\lambda\langle b\rangle)$ $a^b=a^{-1},\ H\langle b\rangle=H\times_f\langle b\rangle$	$12+(3\cdot a_H\cdot 2^{-1})$
	8	$(C_3\times Z)\times_\lambda C_8=(\langle a\rangle\times Z)\times_\lambda\langle b\rangle$ $a^b=a^{-1}$, $C_Z(b^2)=1,\ Z=1$ or $H,\ z^{b^4}=z\ \forall z\in Z$	$12+(3\cdot a_Z\cdot 2^{-1})$
	9	$(C_{13}\times H)\times_\lambda C_4=(\langle a\rangle\times H)\times_\lambda\langle b\rangle$ $a^b=a^{-1}$, $H\langle b\rangle=H\times_f\langle b\rangle$	$16+(13\cdot a_H\cdot 4^{-1})$
	9	$C_3\times(H\times_f C_4)$	$12+(3\cdot a_H\cdot 4^{-1})$
D_{10}	6	$C_2^4\times_\lambda D_{10}=\langle x,y,z,w\rangle\times_\lambda\langle a,b\rangle$ $a^b=a^{-1},\ x^a=y$, $y^a=z,\ z^a=w,\ w^a=xyzw=y^b,\ z^b=w,\ w^b=z,\ x^b=x$	10
	5	SL(2,3)	7
	6	$(C_4\times C_4)\times_f C_3$	8
	8	$(C_2\times C_2\times Q_8)\times_\lambda C_3=(\langle x,y\rangle\times\langle a,b\rangle)\times_\lambda\langle c\rangle$ $x^c=y,\ y^c=xy,\ a^c=b,\ b^c=ab$	12
A_4	8	$(C_3\times C_3\times C_3)\times_\lambda A_4=(\langle x,y,z\rangle)\times_\lambda\langle a_1,a_2,b\rangle$ $x^{a_1}=x=z^b,\ y^{a_1}=y^{-1},\ z^{a_1}=z^{-1}=z^{a_2},\ y^{a_2}=y$, $x^{a_2}=x^{-1},\ x^b=y,\ y^b=z$	13
		$\mathrm{Hol}(2^5\Gamma_2 h, C_3)$	12
C_5	8	$C_2\times(Y\times_f C_5)$	$10+(2\cdot a_H\cdot 5^{-1})$
	7	$(C_3\times H)\times_\lambda Q_8=(\langle x\rangle\times H)\times_\lambda\langle a,b\rangle$ $x^a=x^{-1}=x^b$ $H\langle a,b\rangle=H\times_f\langle a,b\rangle$	$9+(3\cdot a_H\cdot 8^{-1})$
Q_8	8	$C_2\times(H\times_f Q_8)$	$10+(a_H\cdot 4^{-1})$
	8	$H\times_\lambda(C_4\times_\lambda C_4)=H\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ $a^b=a^{-1}$, $H\langle a\rangle=H\times_f\langle a\rangle,\ H\langle b\rangle=H\times_f\langle b\rangle,\ H\langle ab\rangle=H\times_f\langle ab\rangle$	$10+(a_H\cdot 4^{-1})$
D_8	6	$(C_3\times C_3)\times_\lambda D_8=\langle x,y\rangle\times_\lambda\langle a,b\rangle$ $a^b=a^{-1},\ x^a=y$, $y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	9
	8	$C_3\times_\lambda Q_{16}=\langle x\rangle\times_\lambda\langle a,b\rangle$ $x^a=x^{-1}=x^b,\ a^b=a^{-1}$	12
	8	$C_3\times_\lambda D_{16}=\langle x\rangle\times_\lambda\langle a,b\rangle$ $x^a=x^{-1}=x^b,\ a^b=a^{-1}$	12
	8	$C_3\times_\lambda SD_{16}=\langle x\rangle\times_\lambda\langle a,b\rangle$ $x^a=x^{-1}=x^b,\ a^b=a^{-1}$	12

TABLE 5 (contd.)

$G/S(G)$	$\alpha(G)$	G	$r(G)$
D_8	8	$(C_5\times C_5)\times_\lambda D_8=\langle x,y\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$, $x^a=y,\ y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	14
	9	$(C_3\times C_3)\times_\lambda D_{16}=\langle x,y\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$ $x^a=y,\ y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	15
	9	$(C_3\times C_3)\times_\lambda Q_{16}=\langle x,y\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$, $x^a=y,\ y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	15
	9	$(C_3\times C_3)\times_\lambda \mathrm{SD}_{16}=\langle x,y\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$ $x^a=y,\ y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	15
	9	$(C_3\times C_3\times C_3)\times_\lambda D_8=\langle x,y,z\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$ $x^a=x,\ x^b=x^{-1},\ y^a=z,\ z^a=y^{-1},\ y^b=y,\ z^b=z^{-1}$	15
D_{14}	—	—	—
$\mathrm{Hol}\,C_5$	8	$C_2^4\times_\lambda \mathrm{Hol}\,C_5=\langle x,y,z,w\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1}$ $x^a=y=w^b,\ y^a=z=y^b,\ z^a=w,\ w^a=xyzw=z^b,\ x^b=x$	11
	8	$C_{25}\times_f C_4$	10
$C_7\times_f C_3$	6	$\mathrm{Hol}(C_2^3, C_7\times_f C_3)$	8
Σ_4	6	$\mathrm{GL}(2,3)$	8
	6	$\mathrm{SL}(2,3)\cdot C_4$	8
	8	$(C_4\times C_4)\times_\lambda \Sigma_3=\langle x,y\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1},\ x^a=y$, $y^a=x^{-1}y^{-1},\ x^b=x^{-1},\ y^b=xy$	10
	8	$C_2^4\times_\lambda \Sigma_3=\langle x,y,z,w\rangle\times_\lambda\langle a,b\rangle \quad a^b=a^{-1},\ x^a=y,\ x^b=x$, $y^a=xy=y^b,\ z^a=w,\ z^b=xz,\ w^a=zw,\ w^b=xyzw$	10
A_5	7	$\mathrm{SL}(2,5)$	9
	7	$C_2^4\times_{\lambda_1} A_5=\langle x,y,z,w\rangle\times_\lambda\langle a,b\rangle \quad a^5=b^3=1,\ (ba)^2=1$, $x^a=y,\ y^a=z,\ z^a=w,\ w^a=xyzw,\ x^b=xy,\ y^b=x$, $z^b=yzw,\ w^b=xz$	9
	9	$C_2^4\times_{\lambda_2} A_5=\langle x,y,z,w\rangle\times_\lambda\langle a,b\rangle \quad x^a=y,\ y^a=z,\ z^a=w$, $w^a=xyzw,\ x^b=yz,\ y^b=x,\ z^b=xy,\ w^b=zw$	12
C_6	5	$H\times_f C_6$	$6+(a_H\cdot 6^{-1})$
	6	$(C_2\times C_2\times H)\times_\lambda C_6=(\langle x,y\rangle\times H)\times_\lambda\langle a\rangle$ $x^a=y,\ y^a=xy,\ H\langle a\rangle=H\times_f\langle a\rangle$	$8+(2\cdot a_H\cdot 3^{-1})$
	7	$(C_3\times H)\times_\lambda C_6=(\langle x\rangle\times H)\times_\lambda\langle a\rangle \quad x^a=x^{-1}$, $H\langle a\rangle=H\times_f\langle a\rangle$	$9+(a_H\cdot 2^{-1})$
	7	$(C_7\times H)\times_\lambda C_6=(\langle x\rangle\times H)\times_\lambda\langle a\rangle \quad x^a=x^2$ $H\langle a\rangle=H\times_f\langle a\rangle$	$10+(7\cdot a_H\cdot 6^{-1})$
	8	$(C_3\times(C_2\times C_2))\times_\lambda C_6=\langle x,y,z\rangle\times_\lambda\langle a\rangle \quad x^3=y^2=z^2=1$, $x^a=x^{-1},\ y^a=z,\ z^a=yz$	12
	8	$(C_3\times C_2\times C_2\times H)\times_\lambda C_6=(\langle x,y,z\rangle\times H)\times_\lambda\langle a\rangle \quad x^3=1$, $y^2=z^2=1,\ x^a=x^{-1},\ y^a=z,\ z^a=yz,\ H\langle a\rangle=H\times_f\langle a\rangle$	$10+2\|H\|$
	9	$(C_{13}\times H)\times_\lambda C_6=(\langle x\rangle\times H)\times_\lambda\langle a\rangle \quad x^a=x^3$, $H\langle a\rangle=H\times_f\langle a\rangle$	$14+(13\cdot a_H\cdot 6^{-1})$
	9	$(C_3\times C_7)\times_\lambda C_6=\langle x,y\rangle\times_\lambda\langle a\rangle \quad x^3=y^7=1,\ x^a=x^{-1}$ $y^a=y^2$	15

TABLE 5 (contd.)

$G/S(G)$	$\alpha(G)$	G	$r(G)$
	9	$(C_3\times C_7\times H)\times_\lambda C_6=(\langle x,y\rangle\times H)\times_\lambda\langle a\rangle \quad x^3=y^7=1,$ $x^a=x^{-1},\ y^a=y^2,\ H\langle a\rangle=H\times_f\langle a\rangle$	$13+(7\|H\|-3)2^{-1}$
	9	$(C_5\times H)\times_\lambda C_6=(\langle x\rangle\times H)\times_\lambda\langle a\rangle \quad x^a=x^{-1}$ $H\langle a\rangle=H\times_f\langle a\rangle$	$12+(5\cdot a_H\cdot 6^{-1})$
	5	$H\times_f \mathrm{DC}_3$	$6+(a_H\cdot 12^{-1})$
DC_3	7	$(C_5\times H)\times_\lambda \mathrm{DC}_3=(\langle x\rangle\times H)\times_\lambda\langle a,b\rangle \quad a^b=a^{-1},$ $x^a=x,\ x^b=x^2,\ H\langle a,b\rangle=H\times_f\langle a,b\rangle$	$9+(5\cdot a_H\cdot 12^{-1})$
	8	$(C_2\times C_2\times H)\times \mathrm{DC}_3=(\langle x,y\rangle\times H)\times_\lambda\langle a,b\rangle$ $a^b=a^{-1},\ x^a=y,\ y^a=xy,\ x^b=x,\ y^b=xy,\ H\langle a,b\rangle=H\times_f\langle a,b\rangle$	$10+(a_H\cdot 3^{-1})$
$C_2\times\Sigma_3$	9	$C_5^2\times_\lambda(C_2\times\Sigma_3)=(\langle x,y\rangle)\times_\lambda(\langle c\rangle\times_\lambda\langle a,b\rangle) \quad x^c=x^{-1},$ $x^b=x,\ y^c=y^{-1},\ x^a=y,\ y^a=x^{-1}y^{-1}=y^b$	14
$C_3^2\times_f C_2$	7	$Q_1\times_{\lambda_1}C_2=\langle a_1,a_2,b\rangle\times_{\lambda_1}\langle c\rangle$ $a_2^b=a_1a_2,\ a_2^c=a_2^{-1},\ b^c=b^{-1},\ a_1^c=a_1=a_1^b$	10
D_{18}	—	—	—
$C_3^2\times_f C_4$	—	—	—
$C_3^2\times_f Q_8$	—	—	—
	9	SL(2,7)	11
PSL(2,7)	9	$\mathrm{Hol}(C_2\times C_2\times C_2)$	11
	9	$C_2^3\cdot\mathrm{PSL}(2,7)$	11

applied to the cases in which $G/S(G)$ is isomorphic to Σ_3, D_{10}, D_{14} or $C_7\times_f C_3$ yields the listed groups (arguing with Lemmas 2.4 and 2.5).

(3) Suppose $G/S(G)=\langle\bar{b}\rangle\simeq C_4$. We have $O_2(S(G))\leqq Z(G)$, $\alpha(G)=2\cdot|C_G(b)\cap S(G)|+r_G(b^2S(G))$ and $L=C_G(b^2)\cap S(G)\trianglelefteq G$. Set $D\trianglelefteq G$ such that $S(G)=L\times D$ and set $L_1=C_G(b)\cap S(G)\trianglelefteq G$. Then $L_1\leqq L$, $r_G(b^2S(G))=(|L_1|+|L|)/2$ and

$$\alpha(G)=2|L_1|+(|L|+|L_1|)/2. \tag{9}$$

Now (9) yields the listed groups for $5\leqq\alpha(G)\leqq 9$.

(4) Suppose $G/S(G)\simeq C_2\times C_2$. We have $O_2(S(G))\leqq Z(G)$ and $C_G(c)\cap S(G)$ is a normal subgroup of G for every $c\in G-S(G)$. We assume two cases:

(i) There is $c\in G-S(G)$ such that $r_G(cS(G))=1$. In this case, $|C_G(c)|=4$, hence necessarily a 2-Sylow subgroup P of G is isomorphic to $C_2 \times C_2$, D_8, or Q_8, and $G = [O_{2'}(S(G))]P$.

If $P = \langle b_1 \rangle \times \langle b_2 \rangle \simeq C_2 \times C_2$, we have $S(G) = L \times K$ with $C_{S(G)}(b_1) = 1$, $L = C_G(b_2) \cap S(G) \trianglelefteq G$ and $K \trianglelefteq G$. Therefore, $r_G(b_2 S(G)) = 1 + (|L| - 1)/2$, $r_G(b_1 b_2 S(G)) = 1 + (|K| - 1)/2$ and

$$\alpha(G) = 2 + (|K| - 1)/2 + (|L| - 1)/2. \tag{10}$$

If $P = \langle a, b \mid a^4 = 1 = b^2,\ a^b = a^{-1} \rangle \simeq D_8$, we can suppose two cases: (1) $|C_G(b)| = 4$, (2) $|C_G(a)| = 4$. If $|C_G(b)| = 4$, then $O_{2'}(S(G)) = S_1 \times S_2$ with $S_1 = C_G(a) \cap O_{2'}(S(G)) \trianglelefteq G$ and $S_2 \trianglelefteq G$. Therefore, $r_G(aS(G)) = |S_1|$, $r_G(abS(G)) = |S_2|$ and

$$\alpha(G) = 1 + |S_1| + |S_2|. \tag{11}$$

If $|C_G(a)| = 4$, then $S(G) = E_1 \times E_2$ with $E_1 = C_G(b) \cap O_{2'}(S(G)) \trianglelefteq G$, and $E_2 \trianglelefteq G$. So, $r_G(abS(G)) = |E_2|$, $r_G(bS(G)) = |E_1|$ and

$$\alpha(G) = 1 + |E_1| + |E_2|. \tag{12}$$

If $P = \langle a, b \rangle \simeq Q_8$, then $S(G) = E_3 \times E_4$ with $E_3 = C_G(a) \cap O_{2'}(S(G))$, $E_4 \trianglelefteq G$ and

$$\alpha(G) = 1 + |E_3| + |E_4|. \tag{13}$$

Now fixing $\alpha(G)$ in the equations (10), (11), (12) and (13), we obtain the groups corresponding to this case.

(ii) We assume that $r_G(cS(G)) \geqq 2$ for each $c \in G - S(G)$. Set $G/S(G) = \langle \bar{b}_1 \rangle \times \langle \bar{b}_2 \rangle$ and $\nabla = (r_G(b_1 S(G)), r_G(b_2 S(G)), r_G(b_1 b_2 S(G)))$. By symmetry, we can suppose that $\{(2,2,2), (2,2,3), (2,2,4), (2,2,5), (2,3,3), (2,3,4), (3,3,3)\}$ is the set of possible values of ∇. From Lemma 2.2, equation (2), we have that $r_G(bS(G)) = 2$ implies $\Delta_b = (8,8)$ or $(6,12)$ and $r_G(bS(G)) = 3$ implies

$$\Delta_b \in \{(6,24,24), (8,16,16), (10,10,20), (12,12,12)\}.$$

Now using (2.7) and distinguishing the cases in which either 2 divides $|S(G)|$ or $|S(G)|$ is odd number, we obtain all the desired groups.

(5) Suppose $G/S(G) \simeq A_4$. We have $\alpha(G) = r_G(a_1 S(G)) + 2|C_G(b) \cap S(G)|$ with $o(\bar{a}_1) = 2$ and $o(\bar{b}) = 3$. If $S(G)$ is 2-group, then $S(G) \leqq Z(N)$, where N is the normal subgroup of G such that $G/N \simeq C_3$. Therefore, $N/Z(N) \leqq C_2 \times C_2$. If N is abelian, then $r_G(a_1 S(G)) = |N|/4$, $\alpha(G) = |N|/4 + 2|C_G(b) \cap S(G)|$ and necessarily $N \simeq (C_4 \times C_4) \times_f C_3$. If N is non-abelian, we have $N/Z(N) \simeq C_2 \times C_2$, $N' = [N, N] \simeq C_2$, $S(G) = N' \times K$ with $K \trianglelefteq G$ and $r_G(a_1 S(G)) =$

$|N|/8$. Therefore, either $G \simeq \mathrm{SL}(2,3)$, or $G \simeq (C_2 \times C_2 \times Q_8) \times_\lambda C_3$. On the other hand, if $O_{2'}(S(G))$ is non-trivial, then since all elements of order 2 of $G/S(G)$ are conjugates, we have $(C_2 \times C_2)^* = \mathrm{Cl}_{\bar{G}}(\bar{a}_1)$ and necessarily $r_G(a_1 S(G)) \geqq 2$. Now, (2.6) and (2.15) yield $G \simeq (C_3 \times C_3 \times C_3) \times_\lambda A_4$ with the relations given in Table 5.

(6) Suppose $G/S(G) = \langle \bar{b} \rangle \simeq C_5$. We have $\alpha(G) = 4 \cdot |C_G(b) \cap S(G)|$ and $L = C_G(b) \cap S(G) \leq G$. So $S(G) = L \times K$ with $K \trianglelefteq G$ and $C_K(b) = 1$. Consequently $\alpha(G) = 8$ and $G \simeq C_2 \times (Y \times_f C_5)$.

(7) Suppose $G/S(G) = \Sigma_4 = [C_2 \times C_2]\Sigma_3 = [\langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle]\langle \bar{b}, \bar{c} \rangle$ with $\bar{a}_1^{\bar{b}} = \bar{a}_2$, $\bar{a}_2^{\bar{b}} = \bar{a}_1 \bar{a}_2$, $\bar{a}_1^{\bar{c}} = \bar{a}_1$, $\bar{a}_2^{\bar{c}} = \bar{a}_1 \bar{a}_2$. Then

$$\alpha(G) = r_G(a_1 S(G)) + r_G(cS(G)) + |C_G(b) \cap S(G)| + |C_G(ca_2) \cap S(G)|,$$

moreover $(\langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle)^* \subseteq \mathrm{Cl}_{\bar{G}}(\bar{a}_1)$. Now arguing as in the case $\bar{G} \simeq A_4$, we obtain the desired groups.

(8) If $G/S(G) \simeq D_8$ or Q_8, then $O_2(S(G)) \leqq Z(G)$ and we have

$$\alpha(G) = r_G(a^2 S(G)) + r_G(abS(G)) + r_G(bS(G)) + |C_G(a) \cap S(G)|$$

if $G/S(G) = \langle \bar{a}, \bar{b} \mid \bar{a}^4 = 1 = \bar{b}^2, \bar{a}^{\bar{b}} = \bar{a}^{-1} \rangle \simeq D_8$ and

$$\alpha(G) = |C_G(a) \cap S(G)| + |C_G(b) \cap S(G)| + |C_G(ab) \cap S(G)| + r_G(a^2 S(G))$$

if $G/S(G) = \langle \bar{a}, \bar{b} \rangle \simeq Q_8$. Thus, $O_2(S(G)) \leqq C_2$, $S(G) = L \times K$ with $L = C_G(a^2) \cap S(G) \leq G$ and $K \trianglelefteq G$. Now we easily obtain the desired groups.

(9) If $G/S(G) \simeq \mathrm{Hol}\, C_5$, then

$$\alpha(G) = |C_G(a) \cap S(G)| + 2|C_G(b) \cap S(G)| + r_G(b^2 S(G))$$

with $o(\bar{a}) = 5$ and $o(\bar{b}) = 4$, $S(G) = L \times D$ with $L = C_G(a) \cap S(G) \trianglelefteq G$ and $D \trianglelefteq G$. Now, we obtain the desired groups fixing the possible values of $\alpha(G)$.

(10) Suppose $G/S(G) \simeq A_5$. We have

$$\alpha(G) = 2|C_G(c) \cap S(G)| + |C_G(b) \cap S(G)| + r_G(a_1 S(G))$$

with $o(\bar{c}) = 5$, $o(\bar{b}) = 3$, $A_4 = \langle \bar{a}_1, \bar{a}_2 \rangle \langle \bar{b} \rangle$ and $(\langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle)^* \subseteq \mathrm{Cl}_{\bar{G}}(\bar{a}_1)$. (2.15) and $r_G(a_1 S(G)) \leqq 6$ imply $O_{2'}(S(G)) = 1$. Set $L = C_G(c) \cap S(G)$. If $L \neq 1$, then necessarily $L \simeq C_2$ and (2.8) implies that there is a $T \trianglelefteq S(G)\langle c \rangle$ such that $S(G) = L \times T$. Consequently 5 divides $|T| - 1$. If $T = 1$, then $G \simeq \mathrm{SL}(2,5)$. Otherwise, we have $|S(G)| = 2^{1+4k}$ for some integer k, hence b does not act f.p.f. over $S(G)$, thus 2 divides $|C_G(b) \cap S(G)|$, $r_G(a_1 S(G)) \leqq 3$ and necessarily $O_2(S(G)) \leqq C_2^4$ from (2.5); this is impossible. So, we can suppose that $C_G(c) \simeq C_5$. Consequently $|S(G)| = 2^{4k}$ for some integer k. By considering the conjugation action of $N_{A_5}(C_5) \simeq D_{10}$ over $S(G)$ we obtain that $k=1$. Finally, as every non-trivial extension of C_2^4 by A_5 splits (cf. [8] p. 79), it follows that either $G \simeq C_2^4 \times_{\lambda_1} A_5$ or $G \simeq C_2^4 \times_{\lambda_2} A_5$.

If $r(G/S(G)) = 6$, then there is $d \in G - S(G)$ such that $r_G(dS(G)) = 1$, because $\alpha(G) \leqq 9$. Now the study of the associated equation (1) is less complicated, because the possible values of $r_G(xS(G))$, $x \in G - S(G)$ are smaller. Further, the proofs are analogous to those we have discussed above using the lemmas from section 2, and are therefore omitted.

THEOREM 4.3.
$\Phi_1 = \{F_{t,1} \mid t \in \mathbf{N}\}$,
$\Phi_2 = \{C_3\} \cup \{F_{t,2} \mid t \in \mathbf{N}\}$,
$\Phi_3 = \{C_4, D_{10}\} \cup \{F_{t,3} \mid t \in \mathbf{N}\}$,
$\Phi_4 = \{C_5, C_6, D_8, Q_8, D_{12}, \mathrm{DC}_3, D_{14}, \Sigma_4, C_7 \times_f C_3, A_5\} \cup \{F_{t,4} \mid t \in \mathbf{N}\}$,
$\Phi_5 = \{C_2 \times C_4, C_3 \times C_3, C_9 \times_f C_2, C_3^2 \times_f C_4, C_3^2 \times_f Q_8, \mathrm{PSL}(2,7)\}$,
$\Phi_6 = \{C_7, Q_{16}, D_{16}, \mathrm{SD}_{16}, C_{13} \times_f C_4, \mathrm{SL}(2,3), D_{22}, D_{20}, C_5 \times_\lambda C_4, \mathrm{Hol}\, C_7,$
$C_{13} \times_f C_3, C_2 \times A_4, C_{11} \times_f C_5, \Sigma_5, \mathrm{PSL}(2,9)\} \cup \{F_{t,5} \mid t \in \mathbf{N}\}$.

PROOF. It is an immediate consequence from (2.17), (2.18), (2.19), (2.20), (3.2), (4.1) and (4.2).

COROLLARY 4.4. *$r(G) = 7$ if and only if G is isomorphic to one of the groups with exactly seven conjugate classes listed in* Table 1.

PROOF. $r(G) = 7$ implies $\beta(G) \geqq r(G) - 6$, hence $G \in \bigcup_{j=1}^{6} \Phi_j$ and (4.4) is deduced from a simple inspection of the groups with 7 conjugate classes that appear in $\bigcup_{j=1}^{6} \Phi_j$.

LEMMA 4.5. *Let G be a non-nilpotent group with $S(G)$ abelian and satisfying $5 \leqq \alpha(G) \leqq 9$ and $r(G/S(G)) = 7$. Then G is isomorphic to one of the following groups:*

(1) $Y \times_f C_7$,

(2) $(C_3 \times C_3) \times_\lambda \mathrm{SD}_{16} = \langle x, y \rangle \times_\lambda \langle a, b \rangle$, $a^b = a^3$, $x^a = y$, $y^a = xy$, $x^b = x$, $y^b = xy^{-1}$ $(r = 9)$,

(3) $H \times_f Q_{16}$,

(4) $H \times_f \mathrm{SL}(2,3)$,

(5) $(C_3 \times C_3) \times_\lambda \mathrm{SL}(2,3) = \langle x, y \rangle \times_\lambda \langle a, b, c \rangle$, $a^c = b$, $b^c = a^{-1}b^{-1}$, $x^a = y$, $y^a = x^{-1}$, $x^b = x^{-1}y$, $y^b = xy$, $x^a = x$, $y^a = x^{-1}y$ $(r = 10)$,

(6) $(C_3 \times C_3 \times H) \times_\lambda \mathrm{SL}(2,3) = (\langle x, y \rangle \times H) \times_\lambda \langle a, b, c \rangle$, $a^c = b$, $b^c = a^{-1}b^{-1}$, $x^c = y$, $y^a = x^{-1}$, $x^b = x^{-1}y$, $y^b = xy$, $x^a = x$, $y^c = x^{-1}y$ *and* $H\langle a, b, c \rangle = H \times_f \langle a, b, c \rangle$ $(r = 10 + (3 \cdot (|H| - 1)8^{-1})$.

PROOF. If $r(G/S(G))=7$, then $\alpha(G)=\Sigma_{i=1}^{6} r_G(x_iS(G))$. Therefore, at least, there exist three x_j such that $r_G(x_jS(G))=1$, because $\alpha(G)\leqq 9$. Writing the associated equations (1) of the groups with 7 conjugate classes, we easily obtain the desired groups.

THEOREM 4.6. $\Phi_7=\{C_8,\ D_8\times C_2,\ Q_3\times C_2,\ C_4\times_\lambda C_4,\ (C_4\times C_2)\times_{\lambda_2} C_2,$ $\mathrm{GL}(2,3), \mathrm{SL}(2,3), C_4, C_{15}\times_f C_2, C_{13}\times_f C_2, C_3\times\Sigma_3, C_{17}\times_f C_4, C_2^4\times_f C_5, C_5^2\times_f Q_8,$ $C_{13}\times_f C_6,\ C_5^2\times_f \mathrm{DC}_3,\ C_5^2\times_f \mathrm{SL}(2,3),\ (C_2\times C_2)\times_\lambda(C_9\times_f C_2),\ (C_2\times$ $C_2)\times_\lambda(C_3^2\times_f C_2),\ C_2^4\times_\lambda\Sigma_3,\ \Sigma_3\times\Sigma_3,\ C_3\times_\lambda D_8,\ C_{12}\times_\lambda C_2,\ C_3\times_\lambda Q_8,\ C_4^2\times_f C_3,$ $\mathrm{Hol}(C_2^3, C_7\times_f C_3),\ C_2^3\times_\lambda C_4,\ C_2\times(C_3^2\times_f C_2),\ M_9,\ \mathrm{PSL}(2,11)\}\cup\{F_{t,6}\mid t\in\mathbf{N}\}.$

PROOF. It follows from (2.17), (2.18), (2.19), (2.20), (3.2), (4.1), (4.2) and (4.5).

COROLLARY 4.7. *$r(G)=8$ if and only if G is isomorphic to one of the groups with exactly eight conjugate classes listed in Table* 1.

PROOF. We argee as in (4.4).

LEMMA 4.8. *Let G be a non-nilpotent group with $S(G)$ abelian such that $5\leqq\alpha(G)\leqq 9$ and $r(G/S(G))=8$. Then G is isomorphic to one of the groups listed in Table* 6.

PROOF. If $r(G/S(G))=8$, then $\alpha(G)=\Sigma_{i=1}^{7} r_G(x_iS(G))$, so at least there are five x_j such that $r_G(x_jS(G))=1$, because $\alpha(G)\leqq 9$. Now, writing the associated equations (1) of the groups with 8 conjugate classes, we easily obtain the listed groups.

THEOREM 4.9. $\Phi_8=\{C_9,\ C_6\times C_2,\ C_{10},\ A_7,\ \mathrm{PSL}(2,13),\ A_5\times C_2,\ C_{19}\times_f C_3,$ $C_7\times_\lambda C_4,\ C_2\times D_{14},\ (C_5\times C_5)\times_f C_2,\ (C_5\times C_5)\times_{f_2} C_4,\ \mathrm{SL}(2,8),\ C_2\times\Sigma_4,\ C_3^2\times_\lambda D_3,$ $(C_2\times C_2)\times\Sigma_3,\ C_2\times DC_3,\ (C_3\times C_5)\times_\lambda C_4,\ C_{19}\times_f C_6,\ C_5\times_\lambda C_8,\ C_2\times\mathrm{Hol}\,C_5,$ $\mathrm{SL}(2,5),\ (C_2\times C_2)\times_\lambda DC_3,\ (C_7\times C_7)\times_f \mathrm{SL}(2,3),\ C_2^4\times_{\lambda_1} A_5,\ P_1\times_f C_3,\ P_2\times_f C_3,$ $\mathrm{PGL}(2,7),\ C_2\times(C_7\times_f C_3),\ (C_7\times C_7)\times_f(\mathrm{SL}(2,3)\cdot C_4),\ \mathrm{Hol}(C_3^2,\mathrm{SD}_{16})\}\cup\{F_{t,7}\mid t\in$ $\mathbf{N}\}.$

PROOF. It follows from (2.17), (2.18), (2.19), (2.20), (3.2), (4.1), (4.2), (4.5) and (4.8).

COROLLARY 4.10. *$r(G)=9$ if and only if G is isomorphic to one of the groups with exactly nine conjugate classes listed in Table* 1.

PROOF. We argue as in (4.4).

LEMMA 4.11. *Let G be a non-nilpotent group with $S(G)$ abelian such that*

TABLE 6

$G/S(G)$	$\alpha(G)$	G	$r(G)$
	7	$H \times_f C_8$	$8+(a_H \cdot 8^{-1})$
C_8	8	$(C_5 \times H) \times_\lambda C_8 = (\langle x\rangle \times H) \times_\lambda \langle a\rangle$, $x^a = x^2$, $H\langle a\rangle = H \times_f \langle a\rangle$	$10+(5 \cdot a_H \cdot 8^{-1})$
	9	$(C_3 \times C_3 \times H) \times_\lambda C_8 = (\langle x, y\rangle \times H) \times_\lambda \langle a\rangle$, $x^a = y$, $y^a = x^{-1}$, $H\langle a\rangle = H \times_f \langle a\rangle$	$12+(9 \cdot a_H \cdot 8^{-1})$
$C_4 \times C_2$	9	$(C_5 \times C_5) \times_\lambda (C_4 \times C_2) = \langle x, y\rangle \times_\lambda \langle a, b\rangle$, $x^a = x^2$, $x^b = x$, $y^a = y^2$, $y^b = y^{-1}$	14
$C_5 \times_\lambda C_4$	7	$H \times_f (C_5 \times_f C_4)$	$8+(a_H \cdot 20^{-1})$
	8	$(C_{11} \times H) \times_\lambda (C_5 \times_\lambda C_4) = (\langle x\rangle \times H) \times_\lambda \langle a, b\rangle$ $a^b = a^{-1}$, $x^a = x^3$, $x^b = x^{-1}$, $H\langle a, b\rangle = H \times_f \langle a, b\rangle$	$10+(11 \cdot a_H \cdot 20^{-1})$
	8	$(C_4 \times C_4) \times_\lambda C_6 = \langle x, y\rangle \times_\lambda \langle a\rangle$, $x^a = y^{-1}$, $y^a = xy$	10
$C_2 \times A_4$	8	$C_2^4 \times_\lambda C_6 = \langle x, y, z, w\rangle \times_\lambda \langle a\rangle$, $x^a = y$, $y^a = xy$, $z^a = yw$, $w^a = xyzw$	10
	8	$C_4^2 \times_{\lambda_1} C_6 = \mathrm{Hol}(2^5\Gamma_4 d, C_3)$	10
	7	$P_1 \times_f C_3$	9
	7	$P_2 \times_f C_3$	9
$C_2^4 \times_f C_3$	9	$((Q_8 Q_8)_{C_2}) \times_\lambda C_3 = \langle a_1, b_1, a_2, b_2\rangle \times_\lambda \langle c\rangle$, $a_i^c = b_i$, $b_i^c = a_i^{-1} b_i^{-1}$, $i = 1,2$	11
GL(2,3)	9	$\mathrm{Hol}(C_3 \times C_3)$	11
$\mathrm{SL}(2,3).C_4$	7	$H \times_f (\mathrm{SL}(2,3) \cdot C_4)$	$8+(a_H \cdot 48^{-1})$
$C_2^3 \times_f C_7$	8	$P_3 \times_f C_7$	10

$5 \leqq \alpha(G) \leqq 9$ and $r(G/S(G)) = 9$. Then G is isomorphic to one of the following groups:

(i) $H \times_f C_9$,

(ii) $H \times_f \mathrm{SL}(2,5)$,

(iii) $Q_1 \times_{\lambda_2} C_2 = \langle a_1, a_2, b\rangle \times_{\lambda_2} \langle c\rangle$ *with* $a_1^c = a_1^{-1}$, $a_2^c = a_2$, $b^c = b^{-1}$ $(r = 10)$,

(iv) $Q_2 \times_\lambda C_2 = \langle a, b\rangle \times_\lambda \langle c\rangle$ *with* $a^b = a^4$, $a^c = a^{-1}$, $b^c = b$ $(r = 10)$,

(v) $Q_1 \times_\lambda (C_2 \times C_2) = \langle a_1, a_2, b\rangle \times_\lambda \langle c, d\rangle$ *with* $a_1^c = a_1^{-1}$, $a_1^d = a_1^{-1}$, $a_2^c = a_2$, $b^c = b^{-1}$, $a_2^d = a_2^{-1}$, $b^d = b$ $(r = 11)$.

PROOF. If $r(G/S(G)) = 9$, we have $\alpha(G) = \Sigma_{i=1}^{8} r_G(x_i S(G))$, therefore at least there are seven x_j such that $r_G(x_j S(G)) = 1$, because $\alpha(G) \leqq 9$. Now we obtain the desired groups from the associated equations (1) for the groups with exactly 9 conjugate classes.

THEOREM 4.12. $\Phi_9 = \{\mathrm{PSL}(3,4), M_{11}, C_4 \times C_2 \times C_2, C_8 \times_\lambda C_2, (C_4 \times C_2) \times_{\lambda_2} C_2, C_{17} \times_f C_2, C_2^4 \times_\lambda C_6, (C_5 \times C_5) \times_{f_3} C_4, (C_5 \times C_5) \times_f C_6, (C_7 \times C_7) \times_f DC_3, P_3 \times_f C_7, (C_7 \times C_7) \times_f Q_{16}, C_2^4 \times_\lambda D_{10}, C_2^4 \times_\lambda (C_2 \times C_2), C_2^4 \times_\lambda (C_3^2 \times_f C_2), Q_1 \times_{\lambda_1} C_2, Q_2 \times_\lambda C_2, Q_1 \times_{\lambda_2} C_2, C_{17} \times_f C_8, C_2^4 \times_\lambda \Sigma_3, C_4^2 \times_{\lambda_1} C_6, C_{25} \times_f C_4, (C_4 \times C_4) \times_{\lambda_2} C_6, \mathrm{Hol}(C_3 \times C_3, \mathrm{SL}(2,3)), (C_{11} \times C_{11}) \times_f \mathrm{SL}(2,5), (C_4 \times C_4) \times_\lambda \Sigma_3\}$.

PROOF. It follows from (2.17), (2.18), (2.19), (2.20), (3.2), (4.1), (4.2), (4.5), (4.8), and (4.11).

COROLLARY 4.13. *$r(G) = 10$ if and only if G is isomorphic to one of the groups listed in Table 2.*

LEMMA 4.14. *Let G be a non-nilpotent group with $S(G)$ abelian such that $5 \leqq \alpha(G) \leqq 9$ and $r(G/S(G)) = 10$. Then G is isomorphic to one of the following groups:*

(i) $H \times_f C_{10}$,

(ii) $H \times_f (C_7 \times_\lambda C_4)$,

(iii) $H \times_f (C_5 \times_\lambda C_8)$,

(iv) $C_2^4 \times_\lambda DC_3 = \langle x_1, x_2, x_3, x_4 \rangle \times_\lambda \langle a, b \rangle$ *with* $a^b = a^{-1}$, $x_1^a = x_1 x_2$, $x_2^a = x_1$, $x_3^a = x_3 x_4$, $x_4^a = x_3$, $x_1^b = x_1$, $x_2^b = x_1 x_2$, $x_3^b = x_2 x_3$, $x_4^b = x_1 x_3 x_4$,

(v) $C_4^2 \times_\lambda DC_3 = \langle z_1, z_2 \rangle \times_\lambda \langle a, b \rangle$ *with* $z_1^a = z_2$, $z_2^a = z_1^{-1} z_2^{-1}$, $z_1^b = z_1^{-1} z_2^2$, $z_2^b = z_1^{-1} z_2$.

PROOF. We have $r(G/S(G)) = \alpha(G) + 1$, hence $|C_G(x)| = |C_{\bar{G}}(\bar{x})|$ for every $x \in G - S(G)$. Now, from a direct inspection of $\Delta_{\bar{G}}$ for $r(\bar{G}) = 10$, we obtain the desired groups.

THEOREM 4.15. $\Phi_{10} = \{\mathrm{PSL}(2,7) \times C_2, \mathrm{Sz}(8), \mathrm{PSL}(2,17), \mathrm{P\Gamma L}(2,8), (A_5 \times C_3) \times_\lambda C_2, C_{11}, C_{12}, Q_1, Q_2, D_{32}, Q_{32}, \mathrm{SD}_{32}, C_4 \times \Sigma_3, C_3 \times D_{10}, C_2 \times (C_9 \times_f C_2), C_9 \times_\lambda C_4, C_2^4 \times_\lambda \mathrm{Hol}\, C_5, C_3 \times A_4, (C_2 \times C_2) \times_\lambda C_9, (C_3 \times C_7) \times_f C_2, (C_3 \times C_3) \times_\lambda C_8, (C_2 \times C_2 \times C_7) \times_f C_3, C_2 \times ((C_3 \times C_3) \times_f C_4), \Sigma_3 \times D_{10}, (C_3 \times C_7) \times_\lambda C_6, (C_2 \times C_2 \times C_7) \times_\lambda C_6, (C_2 \times C_2 \times Q_8) \times_\lambda C_3, (C_3 \times C_3 \times C_3) \times_\lambda C_4, C_2 \times (C_3^2 \times_f Q_8), C_3 \times_\lambda C_8, (C_3 \times C_3 \times C_3) \times_\lambda Q_8, (C_3 \times C_3) \times_\lambda (C_4 \times_\lambda C_4), C_3 \times_\lambda Q_{16}, C_3 \times_\lambda D_{16}, C_3 \times_\lambda \mathrm{SD}_{16}, C_2^4 \times_\lambda \mathrm{DC}_3, (C_3 \times (C_2 \times C_2)) \times_\lambda C_6, C_{19} \times_f C_2, (C_5 \times C_5) \times_f C_3, ((Q_8 Q_8)_{C_2}) \times_\lambda C_3, C_{29} \times_f C_4, C_{31} \times_f C_5, C_{31} \times_f C_6, C_{19} \times_f C_9, (C_5 \times C_5) \times_f C_8, C_{29} \times_f C_7, Q_1 \times_\lambda (C_9 \times_f C_2), C_4^2 \times_\lambda \mathrm{DC}_3, \mathrm{Hol}(C_3 \times C_3), \mathrm{SL}(2,7), \mathrm{Hol}(C_2 \times C_2 \times C_2), C_2^3 \cdot \mathrm{PSL}(2,7), \mathrm{PGL}(2,9), \Sigma_6\} \cup \{F_{t,8} \mid t \in \mathbf{N}\} \cup 2^5\Gamma_6 \cup 2^5\Gamma_7$.

PROOF. It follows from (2.17), (2.18), (2.19), (2.20), (3.2), (4.1), (4.2), (4.5), (4.8), (4.11) and (4.14).

COROLLARY 4.16. *$r(G) = 11$ if and only if G is isomorphic to one of the groups listed in Table 3.*

COROLLARY 4.17. *$r(G) = 12$ and $\beta(G) > 1$ if and only if G is isomorphic to one of the groups listed in Table 4.*

COROLLARY 4.18. *If $r(G) = 13$, then $\beta(G) \leqq 2$.*

COROLLARY 4.19. *$r(G) = 14$ and $\beta(G) > 3$ iff $G \simeq C_5^2 \times_f C_2$ or $G \simeq C_7^2 \times_f C_6$.*

COROLLARY 4.20. *Set $n \in \mathbf{N}$, $n \geqq 15$. Then $r(G) = n$ and $\beta(G) = n - a$ with $1 \leqq a \leqq 10$, if and only if $G \in \{F'_{t_1,1}, F'_{t_2,2}, F'_{t_3,3}, F'_{t_4,4}, F'_{t_5,5}, F'_{t_6,6}, F'_{t_7,7}, F'_{t_8,8}\}$ with $t_1 = \log_2 n$, $t_2 = \log_3(2n-3)$, $t_3 = (\log_2(3n-8))/2$, $t_4 = \log_5(4n-15)$, $t_5 = \log_7(6n-35)$, $t_6 = (\log_2(7n-48))/3$, $t_7 = (\log_3(8n-63))/2$, $t_8 = \log_{11}(10n-99)$, and where $F'_{t,i}$ denotes $F_{t,i}$ if t is a natural number, and is otherwise dropped from the list.*

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